Robust $H_{\infty}$ control for standard discrete-time singularly perturbed systems

J. Dong and G.-H. Yang

Abstract: The state feedback $H_{\infty}$ control problem for standard discrete-time singularly perturbed systems with polytopic uncertainties is considered. Two methods for designing $H_{\infty}$ controllers are given in terms of solutions to a set of linear matrix inequalities, where one of them is with the consideration of improving the upper bound of singular perturbation parameter $\varepsilon$. Moreover, a method of evaluating the upper bound of singular perturbation parameter $\varepsilon$ with meeting a prescribed $H_{\infty}$ performance bound requirement is also given. Numerical examples are given to illustrate the effectiveness of the proposed methods.

1 Introduction

Slow and fast dynamic phenomena often occur in many physical systems. In state-space framework, such systems are commonly modelled using the mathematical description of singular perturbations, with a small parameter $\varepsilon$ being exploited to determine the degree of separation between slow and fast parts of dynamical systems. In the past three decades, singularly perturbed systems have been intensively studied by many researchers (see [1–9] and the references therein).

On account of a very small singular perturbation parameter $\varepsilon$, the analysis and synthesis approaches for normal systems often lead to ill-conditioned results. Therefore a so-called reduction technique is usually adopted for singularly perturbed systems [10], which is a two-step design methodology. First, through the separate stabilisation of two lower dimensional subsystems in two different time scales, then a composite stabilising controller is synthesised from separate stabilising controllers of the two subsystems, where the controller could be determined without the knowledge of the small singular perturbation parameter.

Recently, control problems of singularly perturbed systems have attracted considerable attentions (see [11–19] and the references therein). Based on the small-gain theorem, [20] studies robust stability problem for continuous- and discrete-time singularly perturbed systems by a unified state-feedback design. In [21], the possibilities of sliding mode control design for synchronous generators are analysed for singularly perturbed systems. Stability analysis and robust controller design of uncertain discrete-time singularly perturbed systems are considered in [22]. Based on Riccati equation approach, unified $H_{\infty}$ approaches are given in [23] and [24]. In recent years, a linear matrix inequality (LMI) technique is developed to solve control problems. Because interior point methods in semidefinite programming can globally and effectively solve LMIs, LMI-based controller design methods are practical and appealing. Based on the LMI technique, state-feedback $H_{\infty}$ control problem for continuous-time linear singularly perturbed systems with norm-bounded uncertainties is studied in [25], and a methodology to design suboptimal controllers in terms of the solution of a set of LMIs is given in [26] and [27].

It is well-known that the accurate knowledge of the stability bound $\varepsilon^*$ of a singularly perturbed system (i.e. the system is stable for $\varepsilon \in [0, \varepsilon^*]$) is very important for applications. The characterisation and computation of the stability bound have attracted considerable efforts for the past over two decades [12, 28–31]. In general, there are two classes of methods to characterise and compute the stability bounds, one is based on the frequency domain transfer functions and another is based on state-space models. Both methods can provide the exact bounds as shown in [29] and [30]. However, the issue of how to improve the bound $\varepsilon^*$ by controller design has not been addressed in the literature, which undoubtedly is very important for the applications of singularly perturbed system theory.

This paper is concerned with the problem of designing $H_{\infty}$ state feedback controller for discrete-time singularly perturbed systems, and how to improve the bound of singular perturbation parameter $\varepsilon$, subject to meeting a prescribed performance bound requirement being considered. First, a method is given for evaluating the upper bound $\varepsilon^*$ of singularly perturbed parameter $\varepsilon$, subject to the considered system with a designed controller to be with a prescribed performance bound for $\varepsilon \in (0, \varepsilon^*)$, where the upper bound $\varepsilon^*$ can be obtained by solving a generalised eigenvalue problem (GEVP) [32]. Secondly, design methods for $H_{\infty}$ controller of discrete-time singularly perturbed systems without the consideration of improving the bound of $\varepsilon$ is given in terms of solutions to LMIs. Finally, an LMI-based approach is presented for simultaneously designing the bound of $\varepsilon$ and $H_{\infty}$ controller for a discrete-time singularly perturbed system.

This paper is organised as follows. Section 2 presents system description and some preliminaries. In Section 3,
two new LMI-based $H_\infty$ controller design methods are presented and a sufficient condition is derived for evaluating the upper bound $\varepsilon^*$ of $\varepsilon$ subject to a prescribed $H_\infty$ performance constraint. The validity of these approaches is illustrated by two numerical examples in Section 4. Finally, Section 5 concludes the paper.

**Notation:** For a symmetric block matrix, $(\ast)$ is used for the blocks induced by symmetry, for example

$$
\begin{bmatrix}
M_{11} & * & * \\
M_{21} & M_{22} & * \\
M_{31} & M_{32} & M_{33}
\end{bmatrix} =
\begin{bmatrix}
M_{11} & M_{12}^T & M_{13}^T \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{bmatrix}
$$

The superscript T stands for matrix transposition and the notation $M^{-1}$ denotes the transpose of the inverse of $M$. $\mathcal{L}_2$ is the Lebesgue space consisting of all discrete-time vector-valued functions that are square-summable over $[0, \infty)$. $\|z\|_2$ denotes the $\mathcal{L}_2$-norm of vector function $z$.

### 2 System description and preliminaries

Consider the following standard discrete-time singularly perturbed system.

**System (1)**

$$
\begin{align*}
x_t(k+1) &= [A_{11} ~ A_{12}] x_t(k) \\
&+ [B_{11} ~ eB_{12}] w(k) + [B_{21} ~ eB_{22}] u(k) \\
z(k) &= [C_{11} ~ C_{12}] x_t(k) + D_{11} w(k) + D_{12} u(k)
\end{align*}
$$

and an equivalent system model can also be given as follows.

**System (2)**

$$
\begin{align*}
x_{\epsilon t}(k+1) &= [A_{11} ~ eA_{12}] x_{\epsilon t}(k) \\
&+ [B_{11} ~ eB_{12}] w(k) + [B_{21} ~ eB_{22}] u(k) \\
z(k) &= [C_{11} ~ C_{12}] x_{\epsilon t}(k) + D_{11} w(k) + D_{12} u(k)
\end{align*}
$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $w(k) \in \mathbb{R}^m$ is the disturbance input, $u(k) \in \mathbb{R}^p$ is the control input and $z(k) \in \mathbb{R}^q$ is the controlled output. The matrices $[A_{11} ~ A_{12}]$, $[B_{11} ~ B_{12}]$, $[B_{21} ~ B_{22}]$, $[C_{11} ~ C_{12}]$, $D_{11}$ and $D_{12}$ are appropriately dimensioned. They belong to the following uncertainty polytope [32]

\[\Omega = \left\{ \left[ \begin{array}{cccc}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{array} \right], \left[ \begin{array}{c}
B_{11} \\
B_{12}
\end{array} \right], \left[ \begin{array}{c}
B_{21} \\
B_{22}
\end{array} \right], \left[ \begin{array}{cccc}
C_{11} & C_{12} \\
D_{11} & D_{12}
\end{array} \right] \right\}\]
The following lemmas will be used in this sequel.

Lemma 1: If there exists a symmetric positive-definite matrix \( P \) such that the following LMIs hold [32]

\[
\begin{bmatrix}
-P + A^T E_P e A + \frac{1}{\gamma} C^T_1 C_1 \\
B^T_1 E_P e A + \frac{1}{\gamma} C^T_1 D_{11} \\
A^T E_P e B_1 + \frac{1}{\gamma} C^T_1 D_{11} \\
B^T_1 E_P e B_1 + \frac{1}{\gamma} D^T_{11} D_{11} - \gamma I
\end{bmatrix} < 0,
\]

for \( \epsilon \in (0, \epsilon^* ] \)

then for each singular perturbation parameter \( \epsilon \in (0, \epsilon^* ] \), the system (1) with \( u(k) = 0 \) is asymptotically stable and its \( H_\infty \) norm is less than \( \gamma \).

Lemma 2: For a given positive scalar \( \epsilon^* \), if the following conditions are satisfied

\[
a > 0 \\
a \epsilon^2 + b \epsilon^* + c < 0 \\
c < 0
\]

(9) (10) (11)

where \( a, b \) and \( c \) are constants, then

\[
a \epsilon^2 + b \epsilon^* + c < 0 \quad \text{for} \quad \epsilon \in [0, \epsilon^*]
\]

(12)

Proof: Consider the following two cases

1. If \( a = 0 \), then from (10) and (11), (12) obviously holds.
2. If \( a > 0 \), we consider the following quadratic function of \( \epsilon \)

\[
y(\epsilon) = a \epsilon^2 + b \epsilon^* + c
\]

(13)

Since \( a > 0 \), \( y(\epsilon) \) is the convex function of \( \epsilon \) [33]. From (10) and (11), it follows that \( y(\epsilon^*) < 0 \) and \( y(0) < 0 \), which further implies that \( y(\epsilon) < 0 \) for \( \epsilon \in [0, \epsilon^*] \), that is, when \( a > 0 \), (12) holds. Thus, the proof is complete.

Based on Lemma 2, we have

Lemma 3: For a given positive scalar \( \epsilon^* \), if the following conditions are satisfied

\[
\begin{align*}
T_1 & \geq 0 \\
e^2 T_1 + e^* T_2 + T_3 & < 0 \\
T_3 & < 0
\end{align*}
\]

(14) (15) (16)

then

\[
e^2 T_1 + e T_2 + T_3 < 0, \quad \text{for} \quad \epsilon \in [0, \epsilon^*]
\]

(17)

Proof: For all non-zero vector \( x(k) \), pre- and post-multiplying (14–16) by \( x^T(k) \) and its transpose, then we have

\[
x^T(k) T_1 x(k) \geq 0
\]

(18)

\[
e^2 x^T(k) T_1 x(k) + e^* x^T(k) T_2 x(k) + x^T(k) T_3 x(k) < 0
\]

(19)

\[
x^T(k) T_3 x(k) < 0
\]

(20)

Denote \( a_{x_1} = x^T(k) T_1 x(k), b_{x_1} = x^T(k) T_2 x(k), c_{x_1} = x^T(k) T_3 x(k) \). Substituting \( a_{x_1}, b_{x_1}, c_{x_1} \) into (18–20), then it follows that

\[
\begin{align*}
a_{x_1} & \geq 0 \\
a_{x_1} \epsilon^2 + b_{x_1} \epsilon^* + c_{x_1} & < 0 \\
c_{x_1} & < 0
\end{align*}
\]

(21) (22) (23)

From (21–23) and applying Lemma 2, we can obtain

\[
e^* x^T(k) T_1 x(k) + e x^T(k) T_2 x(k) + x^T(k) T_3 x(k) < 0,
\]

for \( \epsilon \in [0, \epsilon^*] \)

which implies that matrix inequality (17) holds. Thus, the proof is complete.

\[
\square
\]

3 Main results

In this section, a method of evaluating the upper bound of singularly perturbed parameter \( \epsilon \), subject to the robust stability of the closed-loop system with meeting \( H_\infty \) performance bound requirement, is presented. Moreover, two sufficient conditions for designing \( H_\infty \) controllers are given.

3.1 Computation of stability bound of \( \epsilon \) with an \( H_\infty \) performance bound requirement

First, the following preliminary lemma is needed.

Lemma 4: If there exists a symmetric matrix

\[
Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}
\]

such that the following LMIs hold

\[
\begin{align*}
\epsilon^* & \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] + \left[ \begin{array}{ccc} Q_{12} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \leq 0 \\
0 & -S - S^T & 0 \\
0 & 0 & -\gamma I
\end{align*}
\]

\[
\begin{align*}
& \left[ \begin{array}{ccc} Q_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] - S - S^T \\
& \begin{bmatrix} A & B \\ C & \mathcal{D}_{11} \end{bmatrix} \leq 0
\end{align*}
\]

(24)

\[
\begin{bmatrix} A & B \\ C & \mathcal{D}_{11} \end{bmatrix} \leq 0
\]

for \( \epsilon \in (0, \epsilon^*] \)
then for each singular perturbation parameter \( \varepsilon \in (0, \varepsilon^*], \), the system (1) with \( u(k) = 0 \) is asymptotically stable and its \( H_\infty \) norm is less than \( \gamma \).

**Proof:** See Appendix A. \( \square \)

Based on Lemma 4, a method of evaluating the upper bound of singular perturbation parameter \( \varepsilon \) with a prescribed \( H_\infty \) performance bound requirement is given in the following theorem.

**Theorem 1:** For a given positive scalar \( \varepsilon^* \), if there exist a symmetric matrix \( Q \), and a matrix \( S \) with

\[
Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix}
\]

satisfying the following LMIs

\[
\begin{bmatrix}
Q_{11} & 0 \\
0 & 0
\end{bmatrix} - S - S^T & 0 & S^T A_i^T & S^T C_i^T \\
0 & -\gamma I & B_{ii}^T & D_{ii}^T
\end{bmatrix} < 0
\]

(25)

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
Q_{22} & 0 & 0 & 0
\end{bmatrix} + \varepsilon^2 \begin{bmatrix}
0 & 0 & 0 & 0 \\
Q_{12} & 0 & 0 & 0
\end{bmatrix} < 0,
\]

(26)

for \( i = 1, \ldots, N \)

then for singular perturbation parameter \( \varepsilon \in (0, \varepsilon^*] \), system (1) with \( u(k) = 0 \) is robustly stable and has an \( H_\infty \) norm less than \( \gamma \).

**Proof:** See Appendix B. \( \square \)

**Remark 2:** Theorem 1 presents a method of estimating the upper bound of singularly perturbed parameter \( \varepsilon \) subject to the robust stability of the closed loop system while satisfying an \( H_\infty \) performance bound requirement. An upper bound of \( \varepsilon \) can be obtained by solving inequalities (25) and (26) for the following optimisation problem

Minimise \( \varepsilon^* \) subject to (25) and (26)

which can be effectively solved using LMI control toolbox [34].

### 3.2 Robust \( H_\infty \) controller design

In this section, two methods of designing robust \( H_\infty \) controllers are given, and one of them is with the consideration of improving the upper bound \( \varepsilon^* \) of singular perturbation parameter \( \varepsilon \).

First, a design method without considering the improvement of the upper bound of singular perturbation parameter \( \varepsilon \), which is expressed by \( \varepsilon \)-independent LMIs, is given as follows.

**Theorem 2:** If there exist a symmetric matrix \( Q \), and matrices \( S \) and \( L \) with

\[
Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}
\]

satisfying the following LMIs

\[
\begin{bmatrix}
Q_{11} & 0 \\
0 & 0
\end{bmatrix} - S - S^T & 0 & 0 & -\gamma I \\
0 & -\gamma I & -Q & 0 \\
A_i S + B_{ii} L & B_{ii} & -Q & 0 \\
C_i S + D_{ii} L & D_{ii} & 0 & -\gamma I
\end{bmatrix} < 0
\]

(27)

then there exists a sufficient small \( \varepsilon^* > 0 \) such that for \( \varepsilon \in (0, \varepsilon^*] \), the closed-loop system (6) with

\[
K = L S^{-1}
\]

is robustly stable and has an \( H_\infty \) norm less than \( \gamma \).

**Proof:** From (28), we have \( L = K S \). Substituting \( K S \) for \( L \), then (27) can be rewritten as follows

\[
\begin{bmatrix}
Q_{11} & 0 \\
0 & 0
\end{bmatrix} - S - S^T & 0 & 0 & -\gamma I \\
0 & -\gamma I & -Q & 0 \\
(A_i + B_{ii} K) S + B_{ii} & -Q & 0 & 0 \\
(C_i + D_{ii} K) S & D_{ii} & 0 & -\gamma I
\end{bmatrix} < 0,
\]

(29)

Then, there exists sufficient small \( \varepsilon^*_i, i = 1, \ldots, N \) such that

\[
\varepsilon^*_i \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} + \varepsilon^*_i \begin{bmatrix}
0 & 0 & 0 & 0 \\
Q_{12}^T & 0 & 0 & 0
\end{bmatrix} < 0
\]

(30)

Let

\[
\varepsilon^* = \min_{i=1, \ldots, N} \varepsilon^*_i
\]

By (30) and (31), it follows that

\[
\varepsilon^* \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
Theorem 3: For a given positive scalar $\varepsilon^*$, if there exist symmetric matrix $Q$, and matrices $S$ and $L$, with

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}$$

satisfying (27) and the following LMIs

$$\varepsilon^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & Q_{22} \\ 0 & 0 \end{bmatrix} 0 0 0$$

$$+ \varepsilon \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22}^T \\ 0 & 0 \\ 0 & 0 \end{bmatrix} - S - S^T *$$

$$+ \begin{bmatrix} Q_{11} & 0 \\ 0 & 0 \end{bmatrix} - S - S^T *$$

$$- \gamma I$$

$$(A_i + B_2K)S \quad B_{1i}$$

$$(C_i + D_{12}K)S \quad D_{11i}$$

$$* \quad * \quad * \quad *$$

$$-Q \quad * \quad *$$

$$0 \quad -\gamma I$$

$$< 0 \quad \text{for } i = 1, \ldots, N \quad (32)$$

By (27), (32) and using the arguments similar to that for Theorem 1, it follows that the closed-loop system (6) is robustly stable and has an $H_\infty$ norm less than $\gamma$. Thus, the proof is complete.

Theorem 2 presents a sufficient condition for designing $H_\infty$ controllers for standard discrete-time singularly perturbed systems. However, the upper bound of singular perturbation parameter $\varepsilon$ is not addressed in the design method. Theorem 3 gives a design method with the consideration of upper bound of $\varepsilon$.

Remark 3: Theorem 3 presents an LMI-based method of designing robust state feedback $H_\infty$ controllers with the consideration of improving the upper bound $\varepsilon^*$, of singular perturbation parameter $\varepsilon$, for standard discrete-time singularly perturbed systems with polytopic uncertainty. The result is also applicable for systems without uncertainty, which is the case for $N = 1$ for system (1). For nominal systems, a method of designing $H_\infty$ controllers is also given based on Riccati equation approach in [4]. In next section, a comparison between the method in [4] and the results given by Theorems 2 and 3 will be done via a numerical example.

4 Examples

In this section, we present two numerical examples to illustrate the effectiveness of these proposed methods, where Example 1 presents a comparison between the method in [4] and the results given by Theorems 2 and 3, and the effectiveness of the robust design methods given by Theorems 2 and 3 is illustrated in Example 2.

Example 1: Consider a standard linear time invariant discrete-time singularly perturbed system described by (1) and (3) with $N = 1$ (nominal case, that is $N = 1$) and

$$A_1 = \begin{bmatrix} -1.3 & -0.8 \\ 0.7 & 0.8 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} 0.8 \\ 0.9 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$$

$$C_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_{111} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad D_{121} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$E_\varepsilon = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

By applying the method in [4] to the example, the obtained controller gain $K_0$ and minimal $H_\infty$ performance $\gamma^*$ are given in Table 1. The controller gain $K_0$ and optimal $H_\infty$ performance $\gamma^*$ obtained by using Theorem 2 are given in Table 2. For the prescribed upper bound $\varepsilon^* = 0.01$ of singular perturbation parameter $\varepsilon$, the controller gain $K_0$ and optimal $H_\infty$
the largest perturbation bounds of computation results, the design given by Theorem 3 guarantees

\[ A_1 = \begin{bmatrix} 0.6 & 1 & -2 & 1 \\ 0 & 2 & 0 & -3 \\ 0.1 & 0 & 0 & 1 \\ -0.1 & 0.1 & 0.2 & 0.4 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} 0.2 \\ 0 \\ 0 \\ 0.2 \end{bmatrix} \]

\[ B_{21} = \begin{bmatrix} 0.5 \\ 1 \\ 0 \\ 0.8 \end{bmatrix}, \quad C_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad D_{111} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

\[ D_{121} = \begin{bmatrix} 0 \\ 0.3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.7 & 1 & -3 & 0.8 \\ 0 & 1 & 0 & -2 \\ 0.1 & 0 & 0.3 & 1 \\ -0.2 & 0.5 & 0.1 & 0.6 \end{bmatrix} \]

\[ B_{12} = \begin{bmatrix} 0 \\ 0.3 \\ 0 \\ 0.1 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0.3 \\ 1 \\ 0 \\ 0.4 \end{bmatrix}, \quad C_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \]

\[ D_{112} = \begin{bmatrix} 0 \\ 1.2 \end{bmatrix}, \quad D_{122} = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

By applying Theorem 2 to the example, the obtained controller gain \( K_d \) and optimal \( H_{\infty} \) performance \( \gamma^* \) are given in Table 5. For the given upper bound \( \epsilon^* = 0.1 \) of singular perturbation parameter \( \epsilon \), the controller gain \( K_e \) and optimal

| Table 1: State feedback gain and \( \gamma^* \) given by [4] |
|-----------------|-----------------|-----------------|
| \( K_a \)       | \( \gamma^* \)   | \( \gamma^* \)   |
| 0.8519          | 0.2544          | 4.4890          |

| Table 2: State feedback gain and \( \gamma^* \) given by Theorem 2 with \( N = 1 \) |
|-----------------|-----------------|-----------------|
| \( K_a \)       | \( \gamma^* \)   | \( \gamma^* \)   |
| 0.8667          | 0.5333          | 1.0568          |

| Table 3: State feedback gain and \( \gamma^* \) given by Theorem 3 with \( N = 1 \) |
|-----------------|-----------------|-----------------|
| \( K_a \)       | \( \gamma^* \)   | \( \gamma^* \)   |
| \( \epsilon^* = 0.01 \) | [0.8643 0.5312] | 1.0618          |

performance \( \gamma^* \) obtained by using Theorem 3 to design controller, are given in Table 3.

From Tables 1–3, it is easy to see that Theorems 2 and 3 give much better performances for the example. For the above obtained controller gains \( K_a, K_b \) and \( K_c \), the upper bounds of singular perturbation parameter \( \epsilon \) for meeting different \( H_{\infty} \) performance requirements can be evaluated by using Theorem 1. The obtained upper bounds of singular perturbation parameter \( \epsilon \) are given in Table 4. By the computation results, the design given by Theorem 3 guarantees the largest perturbation bounds of \( \epsilon \).

Example 2: Consider a standard linear time invariant discrete-time singularly perturbed system described by (1) and (3) with \( N = 2 \) and

| Table 4: Upper bounds of \( \epsilon \) |
|-----------------|-----------------|-----------------|
| \( K_a \)       | \( K_b \)       | \( K_c \)       |
| \( \gamma = 1.062 \) | infeasible      | 0.0043          |
| \( \gamma = 1.063 \) | infeasible      | 0.0056          |
| \( \gamma = 1.064 \) | infeasible      | 0.0069          |

\[ \epsilon = 0.1 \]

| Table 5: State feedback gain and \( \gamma^* \) given by Theorem 2 |
|-----------------|-----------------|-----------------|
| \( K_d \)       | \( \gamma^* \)   | \( \gamma^* \)   |
| \( [-0.0795 -1.9342 0.3407] \) | 1.3013          | 1.4137          |

| Table 6: State feedback gain and \( \gamma^* \) given by Theorem 3 |
|-----------------|-----------------|-----------------|
| \( K_a \)       | \( \gamma^* \)   | \( \gamma^* \)   |
| \( \epsilon^* = 0.1 \) | [-0.0732 -1.9184 0.3876] | 1.3845          |
| \( \epsilon^* = 0.1 \) | [-0.0732 -1.9184 0.3876] | 1.3845          |
| \( \epsilon^* = 0.1 \) | [-0.0732 -1.9184 0.3876] | 1.3845          |

Table 7: Upper bounds of \( \epsilon \)

<table>
<thead>
<tr>
<th>( K_d )</th>
<th>( K_a )</th>
<th>( K_b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma = 1.31 )</td>
<td>0.0023</td>
<td>0.1043</td>
</tr>
<tr>
<td>( \gamma = 1.32 )</td>
<td>0.0051</td>
<td>0.1075</td>
</tr>
<tr>
<td>( \gamma = 1.33 )</td>
<td>0.0078</td>
<td>0.1108</td>
</tr>
</tbody>
</table>

From Tables 5 and 6, it can be seen that the design given by Theorem 2 can achieve a slightly better performance than that by Theorem 3. However, from Table 7, the design given by Theorem 3 can guarantee a bigger upper bound of singular perturbation parameter \( \epsilon \).

5 Conclusion

In this paper, two methods for designing \( H_{\infty} \) controllers for standard discrete-time singularly perturbed systems with polytopic uncertainties are given in terms of solutions to a set of LMIs, where, one of them is with the consideration of improving the upper bound of singular perturbation parameter \( \epsilon \). Moreover, a method of evaluating the upper bound of singular perturbation parameter \( \epsilon \) to meet a prescribed \( H_{\infty} \) performance bound requirement is also given. The numerical examples have shown the effectiveness of the proposed methods.

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7 References


8 Appendix A: proof of Lemma 4

Proof: (24) can be rewritten as follows

\[
\begin{bmatrix}
E_s Q E_s - S - S^T & 0 & S^T A & S^T C_1^T \\
0 & -\gamma I & B_1^T & D_{11}^T \\
A S & B_1 & -Q & 0 \\
C_1 S & D_{11} & 0 & -\gamma I
\end{bmatrix} < 0,
\]

for \( e \in (0, e^* \] (35)

From (35), we have \( Q > 0, S + S^T > 0 \). Pre- and post-multiplying (35) by

\[
\begin{bmatrix}
S^T & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I
\end{bmatrix}
\]

and its transpose, then it follows that

\[
S^T E_s Q E_s S^{-1} - S^{-1} - S^T - 0
\]

\[
0
\]

\[
-\gamma I
\]

\[
A
\]

\[
B_1
\]

\[
C_1
\]

\[
D_{11}
\]

\[
\begin{bmatrix}
A^T & C_1^T \\
B_1^T & D_{11}^T
\end{bmatrix} < 0,
\]

for \( e \in (0, e^* \] (36)

Let \( P = E_s Q E_s \), therefore

\[
P > 0
\]

(37)

then

\[
(S^T - P) P^{-1} (S^{-1} - P) \geq 0
\]

which implies that

\[
S^T P^{-1} S^{-1} - S^{-1} - S^T \geq -P
\]
combining it with (36), we can obtain
\[
\begin{bmatrix}
-P & 0 & \mathcal{A}^T & \mathcal{C}_1^T \\
0 & -\gamma I & \mathcal{B}_1^T & \mathcal{D}_{11}^T \\
\mathcal{A} & \mathcal{B}_1 & -E_\varepsilon^T P^{-1} E_\varepsilon & 0 \\
\mathcal{C}_1 & \mathcal{D}_{11} & 0 & -\gamma I \\
\end{bmatrix} < 0,
\]
for \( \varepsilon \in (0, \varepsilon^*] \) \hspace{1cm} (38)

Applying Schur complement lemma to (38), we have
\[
\begin{bmatrix}
-P + \mathcal{A}^T E_\varepsilon P E_\varepsilon \mathcal{A} + (1/\gamma) \mathcal{C}_1^T \mathcal{C}_1 \\
\mathcal{B}_1^T E_\varepsilon P E_\varepsilon \mathcal{A} + (1/\gamma) \mathcal{D}_{11}^T \mathcal{C}_1 \\
\end{bmatrix} < 0,
\]
for \( \varepsilon \in (0, \varepsilon^*] \)
\[
\begin{bmatrix}
\mathcal{A}^T E_\varepsilon P E_\varepsilon \mathcal{B}_1 + (1/\gamma) \mathcal{C}_1^T \mathcal{D}_{11} \\
\mathcal{B}_1^T E_\varepsilon P E_\varepsilon \mathcal{B}_1 + (1/\gamma) \mathcal{D}_{11}^T \mathcal{D}_{11} - \gamma I \\
\end{bmatrix} < 0,
\]
for \( \varepsilon \in (0, \varepsilon^*] \) \hspace{1cm} (39)

Then applying Lemma 1 to (39), we have that, for each singular perturbation parameter \( \varepsilon \in (0, \varepsilon^*] \), the system (1) with \( u(k) = 0 \) is asymptotically stable and its \( H_\infty \) norm is less than \( \gamma \). Thus, the proof is complete. \( \square \)

9 Appendix B: proof of Theorem 1

Proof: From (26), we have \( Q > 0 \), then \( Q_{22} > 0 \). Let
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & Q_{22} & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & Q_{12} & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
Q_{11} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
0 & -S - S^T & 0 \\
S^T \mathcal{A} & S^T \mathcal{C}_1 \\
\mathcal{A}^T S & \mathcal{B}_1 & -Q \\
\mathcal{C}_1^T S & \mathcal{D}_{11} & 0 \\
\end{bmatrix}
\]

Then \( \mathcal{T}_1 \geq 0 \). Moreover, from (25) and (26), it follows that \( \mathcal{T}_2 < 0 \), and \( \mathcal{T}_3 < 0 \). By using Lemma 3, we have
\[
\mathcal{T}_1 + \varepsilon^2 \mathcal{T}_2 + \varepsilon \mathcal{T}_3 < 0, \quad \text{for} \quad \varepsilon \in [0, \varepsilon^*], \quad i = 1, \ldots, N \hspace{1cm} (40)
\]

Multiplying (40) by \( \alpha_i, i = 1, \ldots, N \) and summing them from \( i = 1 \) to \( i = N \), then we have
\[
\mathcal{T}_1 + \varepsilon \mathcal{T}_2 + \mathcal{T}_3 < 0, \quad \text{for} \quad \varepsilon \in [0, \varepsilon^*] \hspace{1cm} (41)
\]

where
\[
\mathcal{T}_3 = \sum_{i=1}^N \alpha_i \mathcal{T}_{3i}
\]

Thus, (41) is same as (24). Then from (41) and Lemma 4, it follows that for singular perturbation parameter \( \varepsilon \in (0, \varepsilon^*] \), the system (1) with \( u(k) = 0 \) is robustly stable and has an \( H_\infty \) norm less than \( \gamma \). Thus, the proof is complete. \( \square \)