Static output feedback $H_{\infty}$ control of a class of nonlinear discrete-time systems

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Abstract

This paper is concerned with the problem of designing robust $H_{\infty}$ static output feedback (SOF) controllers for a class of discrete time nonlinear systems, which are represented by T–S fuzzy models. By considering the properties of system output matrices, a method for $H_{\infty}$ SOF control synthesis is developed in terms of solutions to a set of linear matrix inequalities (LMIs). In contrast to the existing techniques for designing SOF controllers, the new proposed method can give less conservative results by removing the constraint that the considered Lyapunov matrix is diagonal. Numerical examples are given to illustrate the effectiveness of the proposed design method.

Keywords: Fuzzy control; Takagi–Sugeno (T–S) fuzzy models; $H_{\infty}$ control; Static output feedback; Linear matrix inequalities (LMIs)

1. Introduction

It is well known that a wide class of nonlinear systems can be approximated by Takagi–Sugeno (T–S) fuzzy models [23], which are locally linear models connected by IF–THEN rules. Due to each sub-model is linear, the conventional linear system theory can be applied to the class of nonlinear control systems, and stability analysis and control synthesis have been widely studied, see [5,9,10,13,15–17,21,25,26,28] an overview paper [22] and a survey paper [7]. However, most of the above-mentioned results are based on the single Lyapunov function approach (i.e., parameter-independent Lyapunov function). Since a common Lyapunov function is used to ensure stability for all subsystems, they can be quite conservative. Then piecewise quadratic Lyapunov functions [6,11,29] and parameter-dependent Lyapunov functions [1,24], $k$-sample variation Lyapunov functions [14] are applied to fuzzy control systems for obtaining less conservative stability analysis and controller design conditions. In particular, by introducing slack variables to separate system matrix and Lyapunov matrix, and based on parameter-dependent Lyapunov functions, control synthesis of linear time-varying systems are considered in [3] and the approach is also applied to the control synthesis of fuzzy systems, see [8] for designing state feedback controllers and [32] for dynamic output feedback control.

Recently, there is rapidly growing interest in $H_{\infty}$ control for T–S fuzzy model, see [1,6,27]. However, most of $H_{\infty}$ designs of fuzzy control systems are under the assumption that the states are available for controller implementations, which is not true in many practical cases. Then the methods for designing $H_{\infty}$ dynamic output feedback controllers are developed in [19,20]. In contrast to dynamic output feedback, static output feedback (SOF) is less expensive for

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controller implementations and more reliable in practice [10]. In particular, [18] provides a sufficient condition for designing robust $H_{\infty}$ SOF controllers by using diagonal structure Lyapunov matrices for uncertain discrete-time T–S fuzzy systems. Moreover, by inserting an equality constrained condition about Lyapunov matrix, sufficient conditions are given in [2] for linear systems and [12] for T–S fuzzy systems. Though these conditions are convex, the requirements for Lyapunov matrix to be with diagonal structure [18] or satisfy an equality condition [2,12] are strict, which might result in conservative designs.

This paper will continue to study the problem of designing $H_{\infty}$ SOF controllers for discrete-time T–S fuzzy systems. The purpose is to develop less conservative conditions for the SOF control synthesis. Based on parameter-dependent Lyapunov functions approaches and consider the properties of system output matrices, a new method for designing $H_{\infty}$ SOF controllers is given and it is shown that the new proposed design method is less conservative than the one in [18]. Though parameter-dependent Lyapunov function approaches have been extensively used for various control problems [3,8,32], in contrast to the existing approaches, the lower triangular structure constraint exploited in this paper is only imposed on a parameter-dependent slack variable, and the Lyapunov matrix is with no structure constraint. That is, the strict constraints on Lyapunov matrices used in [2,12,18] are removed, then the new method is very helpful for giving less conservative results for SOF control synthesis. The result in this paper is also the extension of the previous one in [4], where we present a sufficient condition for designing SOF $H_{\infty}$ controllers for T–S fuzzy systems by introducing a parameter-independent variable. In contrast to the result of [4], the condition with an introduced parameter-dependent variable in this paper can give less conservative results than the one in [4].

The paper is organized as follows. In Section 2, system description and some preliminaries are given. In Section 3, an linear matrix inequality (LMI)-based SOF controller design method is proposed. Section 4 presents two numerical examples to illustrate the effectiveness of the proposed design method. Finally, Section 5 concludes the paper.

**Notation:** The superscript $T$ stands for matrix transposition and the notation $M^{-1}$ denotes the transpose of the inverse matrix of $M$. The symbol $*$ within a square matrix represents the symmetric entries.

### 2. Problem statement and preliminaries

Consider the following uncertain T–S fuzzy system:

$$
\begin{align*}
    x(k + 1) &= \sum_{i=1}^{r} z_i(k)((A_i + \Delta A)x(k) + (B_{1i} + \Delta B_1)w(k) + (B_{2i} + \Delta B_2)u(k)) \\
    z(k) &= \sum_{i=1}^{r} z_i(k)((C_{1i} + \Delta C_1)x(k) + (D_{1i} + \Delta D_1)w(k) + (D_{2i} + \Delta D_2)u(k)) \\
    y(k) &= C_2x(k)
\end{align*}
$$

(1)

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the control input, $z(k) \in \mathbb{R}^i$ is the controlled output, $y(k) \in \mathbb{R}^p$ is the measurable output, $w(k) \in \mathbb{R}^q$ is unknown but energy-bounded disturbance input (i.e., $w(k) \in l_2[0, \infty]$) and

$$
\begin{pmatrix}
    \Delta A & \Delta B_1 & \Delta B_2 \\
    \Delta C_1 & \Delta D_1 & \Delta D_2
\end{pmatrix}
= 
\begin{pmatrix}
    M\Delta(k)(N_1 & N_2 & N_3) \\
    M_\zeta A_\zeta(k)(N_\zeta_1 & N_\zeta_2 & N_\zeta_3)
\end{pmatrix}

A_\zeta^T(k)A_\zeta(k) \leq I

A_\zeta^T(k)I \leq I
$$

Note that the measurable output $y(k)$ are assumed to be linearly dependent on the state $x(k)$. The same assumption is also considered in [10,18]. Moreover, we also assume that $C_2$ is of full row rank through the paper (the same assumption can be found in [2,18]) and let invertible matrix $T$ such that

$$
C_2T = [I \ 0]
$$

(2)

**Remark 1.** For each $C_2$, the corresponding $T$ generally is not unique. A special $T$ can be obtained by the following formula:

$$
T = [C_2^T(C_2C_2^T)^{-1} C_2^\perp]
$$

(3)

where $C_2^\perp$ denotes an orthogonal basis for the null space of $C_2$. 
In this paper, the concept of parallel distributed compensation (PDC) is used to design fuzzy controller, i.e., the designed fuzzy controller shares the same fuzzy sets with the fuzzy model in the premise parts, more details can found in [25]. For the fuzzy model (1), the following SOF controller is exploited:

\[ u(k) = \sum_{i=1}^{r} \alpha_i(k) K_i y(k) \]  

Combining (1) and (4), the closed-loop system is obtained as

\[
x(k + 1) = (A(\alpha(k)) + B_2(\alpha(k))K(\alpha(k))C_2 + M\Delta(k)(N_1 + N_3 K(\alpha(k))C_2))x(k)
+ (B_1(\alpha(k)) + M\Delta(k)N_2)w(k)
\]

\[
z(k) = (C_1(\alpha(k)) + D_2(\alpha(k))K(\alpha(k))C_2 + M_\varepsilon A_2(k)(N_{\varepsilon 1} + N_3 K(\alpha(k))C_2))x(k)
+ (D_1(\alpha(k)) + M_\varepsilon A_2(k)N_{\varepsilon 2})w(k)
\]  

where

\[
A(\alpha(k)) = \sum_{i=1}^{r} \alpha_i(k)A_i, \quad B_1(\alpha(k)) = \sum_{i=1}^{r} \alpha_i(k)B_{1i}, \quad B_2(\alpha(k)) = \sum_{i=1}^{r} \alpha_i(k)B_{2i}
\]

\[
C_1(\alpha(k)) = \sum_{i=1}^{r} \alpha_i(k)C_{1i}, \quad D_1(\alpha(k)) = \sum_{i=1}^{r} \alpha_i(k)D_{1i}, \quad D_2(\alpha(k)) = \sum_{i=1}^{r} \alpha_i(k)D_{2i}
\]

\[
K(\alpha(k)) = \sum_{i=1}^{r} \alpha_i(k)K_i
\]

In this paper, we consider \( H_\infty \) control problem.

**Definition 2** (Zhou et al. [32]). Given a scalar \( \gamma > 0 \), the closed-loop system (7) is said to be asymptotically stable with an \( H_\infty \) performance \( \gamma \) if it is asymptotically stable in the large when \( w(k) \equiv 0 \), and under zero initial condition, \( \|z\|_2 < \gamma \|w\|_2 \) for all nonzero \( \{w(k)\} \in l_2[0, \infty) \).

The following preliminaries will be used in the sequel. Let \( x = T\bar{x} \), then we can obtain the following transformed closed-loop uncertain fuzzy system:

\[
\bar{x}(k + 1) = (A(\alpha(k)) + B_2(\alpha(k))K(\alpha(k))\tilde{C}_2 + \tilde{M}\Delta(k)(\tilde{N}_1 + N_3 K(\alpha(k))\tilde{C}_2))\bar{x}(k)
+ (B_1(\alpha(k)) + \tilde{M}\Delta(k)e_i)\bar{w}(k)
\]

\[
z(k) = (C_1(\alpha(k)) + D_2(\alpha(k))K(\alpha(k))\tilde{C}_2 + M_\varepsilon A_2(k)(\tilde{N}_{\varepsilon 1} + N_3 K(\alpha(k))\tilde{C}_2))\bar{x}(k)
+ (D_1(\alpha(k)) + M_\varepsilon A_2(k)N_{\varepsilon 2})\bar{w}(k)
\]  

The transformed system matrices are defined as

\[
\tilde{A}(\alpha(k)) = T^{-1}A(\alpha(k))T, \quad \tilde{B}_1(\alpha(k)) = T^{-1}B_1(\alpha(k))
\]

\[
\tilde{B}_2(\alpha(k)) = T^{-1}B_2(\alpha(k)), \quad \tilde{C}_1(\alpha(k)) = C_1(\alpha(k))T, \quad \tilde{C}_2 = C_2T = [I \ 0]
\]

\[
\tilde{M} = T^{-1}M, \quad \tilde{N}_1 = N_1T, \quad \tilde{N}_{\varepsilon 1} = N_{\varepsilon 1}T
\]

The following results is from [18], which gives a solution to the above problem.

**Lemma 3** (Lo and Lin [18]). Given a prescribed \( H_\infty \) performance index \( \gamma > 0 \), if there exist a symmetric matrix \( X \), and matrices \( L_i, \ 1 \leq i \leq r \), with

\[
X = \begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} \end{bmatrix}, \quad L_i = [L_{i0} \ 0]
\]

satisfying the following LMIs:

\[
M_{ii} < 0, \quad 1 \leq i \leq r
\]  

(9a)
\[ \frac{1}{r-1} M_{ii} + \frac{1}{2}(M_{ij} + M_{ji}) < 0, \quad 1 \leq i \neq j \leq r \]  

(9b)

where

\[
M_{ij} = \begin{bmatrix}
-X & * & * & * & * & * \\
0 & -e_{1}I & * & * & * & * \\
\tilde{A}_{i} X + \tilde{B}_{2i} L_{j} & -X & * & * & * & * \\
\tilde{C}_{i} X + D_{2i} L_{j} & D_{1i} & 0 & -I & * & * & * \\
0 & 0 & 0 & e_{1} \tilde{M}^{T} & 0 & -e_{1} I & * & * \\
\tilde{N}_{1} X + N_{3} L_{j} & N_{2} & 0 & 0 & 0 & 0 & -e_{2} I & * \\
0 & 0 & 0 & 0 & e_{2} M_{2}^{T} & 0 & 0 & 0 & -e_{2} I \\
\tilde{N}_{1} X + N_{3} L_{j} & N_{2} & 0 & 0 & 0 & 0 & -e_{2} I \\
\end{bmatrix}
\]

then the closed-loop system (5) with the fuzzy controller (4) with

\[
K_{i} = L_{i0} X_{11}^{-1}, \quad 1 \leq i \leq r.
\]  

(10)

is asymptotically stable and with an \( H_{\infty} \) performance bound \( \gamma \).

**Remark 4.** By imposing a diagonal structure constraint on Lyapunov matrix, Lemma 5 [18] provides a convex sufficient condition for solving the SOF control problem. Moreover, by inserting an equality constrained condition about Lyapunov matrix, sufficient conditions are given in [2] for linear systems and [12] for T–S fuzzy systems. On the account of the constraints on Lyapunov matrices (i.e., the equality condition constraint [2,12] and diagonal structure constraint [18] about Lyapunov matrix), these conditions might give conservative results, see Example 14 in Section 4. On the other hand, by using the technique of introducing slack variables, the parameter-dependent Lyapunov function approaches have been used for linear uncertain systems [3] and T–S fuzzy systems [32]. In this paper, a technique of introducing a parameter-dependent slack variable with lower triangular structure and parameter-dependent Lyapunov functions are exploited to derive a new method for designing \( H_{\infty} \) SOF controllers, and it is shown that the new proposed design method is less conservative than Lemma 3. In particular, the strict constraints on Lyapunov matrices used in [2,12,18] are removed.

### 3. Main results

In this section, a new LMI-based \( H_{\infty} \) SOF controller design method for discrete-time uncertain fuzzy systems will be given. Before giving the main result, the following two lemmas are needed.

**Lemma 5** (Xie [30], Yoneyama [31]). Let \( E \) and \( F \) are proper dimensional constant matrix and \( A^{T} A \leq I \), then for any \( \varepsilon > 0 \)

\[
\begin{bmatrix}
0 & 0 & E^{T} A^{T} M^{T} \\
0 & 0 & F^{T} A^{T} M^{T} \\
M \Delta E & M \Delta F 
\end{bmatrix} \leq \begin{bmatrix}
\frac{1}{\varepsilon} E^{T} E & \frac{1}{\varepsilon} E^{T} F & 0 \\
\frac{1}{\varepsilon} F^{T} E & \frac{1}{\varepsilon} F^{T} F & 0 \\
0 & 0 & \varepsilon M M^{T} 
\end{bmatrix}
\]

**Lemma 6.** If \( S + S^{T} > 0 \) for a square matrix \( S \), then \( S \) is invertible.

**Proof.** The proof is easily obtained and omitted. \( \square \)

Based on Lemmas 5 and 6, and by introducing parameter-dependent slack variables \( S(\sigma(k), \sigma(k + 1)) \) with lower triangular structure to separate system matrix \( A(\sigma(k)) \) and Lyapunov matrix \( P(\sigma(k)) \), a sufficient condition for designing SOF controllers is given in the following theorem.

**Theorem 7.** Given a prescribed \( H_{\infty} \) performance index \( \gamma > 0 \), if there exist symmetric matrices positive \( Q_{i} > 0 \), and matrices \( S_{il}, L_{i}, 1 \leq i, l \leq r \), with

\[
S_{il} = \begin{bmatrix}
S_{0} & 0 \\
S_{21i} & S_{22i}
\end{bmatrix}, \quad L_{i} = [L_{i0} \ 0]
\]

(11)
satisfying the following LMIs:
\[
\sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{l=1}^{r} \alpha_i(k) \alpha_j(k) \alpha_l(k) A_{ijl} < 0
\]  
(12)

where

\[
A_{ijl} = \begin{bmatrix}
Q_i - S_{ij} - S_{il}^T & * & * & * & * & * \\
0 & -\gamma^2 I & * & * & * & * \\
\bar{A}_i S_{jl} + \bar{B}_j L_j & \bar{B}_{ji} - Q_i & * & * & * & * \\
0 & 0 & \bar{e}_1 \bar{M}^T & 0 & -\bar{e}_1 I & * & * \\
\bar{N}_1 S_{jl} + N_2 L_j & N_2 & 0 & 0 & 0 - \bar{e}_1 I & * & * \\
0 & 0 & 0 & \bar{e}_2 \bar{M}^T & 0 & 0 & -\bar{e}_2 I \\
\bar{N}_{z1} S_{jl} + N_{z2} L_j & N_{z2} & 0 & 0 & 0 & 0 & 0 & -\bar{e}_2 I \\
\end{bmatrix}
\]  
(13)

then the closed-loop system (5) with the fuzzy controller (4) with

\[
K_i = L_i j_0 S^{-1}_0, \quad 1 \leq i \leq r.
\]  
(14)

is asymptotically stable and with an \( H_\infty \) performance bound \( \gamma \).

**Proof.** Let \( Q(z(k)) = \sum_{i=1}^{r} \alpha_i(k) Q_i \). From \( Q_i > 0, \ 1 \leq i \leq r \), then there exist scalar values \( \bar{e}_i > 0, \ 1 \leq i \leq r \), such that \( Q_i > \bar{e}_i I > 0, \ 1 \leq i \leq r \).

Let

\[
\bar{e} = \min_{1 \leq i \leq r} \{ \bar{e}_i \}
\]

then \( Q_i > \bar{e}_i I \geq \bar{e} I \)

Let \( Q(z(k)) = \sum_{i=1}^{r} \alpha_i(k) Q_i \), then

\[
Q(z(k)) = \sum_{i=1}^{r} \alpha_i(k) Q_i > \sum_{i=1}^{r} \alpha_i(k) \bar{e} I = \bar{e} I > 0
\]

Let \( P(z(k)) = Q^{-1}(z(k)) \). Choose a candidate Lyapunov function

\[
V(k) = \tilde{x}^T(k) P(z(k)) \tilde{x}(k)
\]

then

\[
\begin{align*}
V(k+1) - V(k) + z^T(k) z(k) - \gamma^2 w^T(k) w(k) \\
= & [(\tilde{A}(z(k)) + \tilde{B}_2(z(k)) K(z(k)) \tilde{C}_2 + \tilde{M}(z(k))(\tilde{N}_1 + N_3 K(z(k)) \tilde{C}_3)) \tilde{x}(k) \\
+ (\tilde{B}_1(z(k)) + \tilde{M}(z(k)) N_2) w(k)]^T P(z(k + 1)) \\
& \times [((\tilde{A}(z(k)) + \tilde{B}_2(z(k)) K(z(k)) \tilde{C}_2 + \tilde{M}(z(k))(\tilde{N}_1 + N_3 K(z(k)) \tilde{C}_3)) \tilde{x}(k) \\
+ (\tilde{B}_1(z(k)) + \tilde{M}(z(k)) N_2) w(k)) - \tilde{x}^T(k) P(z(k)) \tilde{x}(k) \\
+ [(\tilde{C}_1(z(k)) + D_2(z(k)) K(z(k)) \tilde{C}_2 + M_\varepsilon A_\varepsilon(k)(\tilde{N}_{z1} + N_{z3} K(z(k)) \tilde{C}_2)) \tilde{x}(k) \\
+ (D_1(z(k)) + M_\varepsilon A_\varepsilon(k) N_{z2}) w(k)]^T \\
& \times [(\tilde{C}_1(z(k)) + D_2(z(k)) K(z(k)) \tilde{C}_2 + M_\varepsilon A_\varepsilon(k)(\tilde{N}_{z1} + N_{z3} K(z(k)) \tilde{C}_2)) \tilde{x}(k) \\
+ (D_1(z(k)) + M_\varepsilon A_\varepsilon(k) N_{z2}) w(k)] - \gamma^2 w^T(k) w(k)
\end{align*}
\]  
(15)

For a clear presentation, denote

\[
A = \tilde{A}(z(k)) + \tilde{B}_2(z(k)) K(z(k)) \tilde{C}_2 + \tilde{M}(z(k))(\tilde{N}_1 + N_3 K(z(k)) \tilde{C}_3)
\]
\[ \begin{align*}
\mathbf{B} &= \hat{B}_1(\alpha(k)) + \bar{M} \Delta(k) N_2 \\
\mathbf{C} &= \bar{C}_1(\alpha(k)) + D_2(\alpha(k)) K(\alpha(k)) \bar{C}_2 + M_2 \Delta_2(k)(\bar{N}_{\varepsilon} + N_{\varepsilon} K(\alpha(k)) \bar{C}_2) \\
\mathbf{D} &= D_1(\alpha(k)) + M_2 \Delta_2(k) N_{\varepsilon} 
\end{align*} \]

then (15) can be rewritten as

\[ V(k + 1) - V(k) = \mathbf{A}^T(\alpha(k + 1)) \mathbf{A}^T(w(k + 1)) P(\alpha(k + 1)) - \mathbf{P}(\alpha(k + 1)) \mathbf{A}^T(\alpha(k + 1)) \mathbf{A} P(\alpha(k)) - \mathbf{P}(\alpha(k)) \mathbf{C}^T \mathbf{C} \mathbf{P}(\alpha(k)) - \mathbf{P}(\alpha(k + 1)) \mathbf{C}^T \mathbf{D} w(k) + w^T(k) \mathbf{D} \mathbf{D} - \gamma^2 I w(k) \]

\[ = \begin{bmatrix} \bar{x}(k) \\ w(k) \end{bmatrix}^T \begin{bmatrix} \mathbf{A}^T P(\alpha(k + 1)) \mathbf{A} - \mathbf{P}(\alpha(k)) + \mathbf{C}^T \mathbf{C} & \mathbf{A}^T P(\alpha(k + 1)) \mathbf{B} + \mathbf{C}^T \mathbf{D} \\
\mathbf{B}^T P(\alpha(k + 1)) \mathbf{A} + \mathbf{D}^T \mathbf{C} & \mathbf{B}^T P(\alpha(k + 1)) \mathbf{B} + \mathbf{D}^T \mathbf{D} - \gamma^2 I \end{bmatrix} \begin{bmatrix} \bar{x}(k) \\ w(k) \end{bmatrix} \]

(17)

On the other hand, from (11) and (14), we have

\[ L_j = [L_{j0} \ 0] = [K_j S_0 \ 0] = [K_j \ 0] \begin{bmatrix} S_0 & 0 \\
S_{21}(\alpha(k), \alpha(k + 1)) & S_{22}(\alpha(k), \alpha(k + 1)) \end{bmatrix} \]

\[ = K_j [I \ 0] S(\alpha(k), \alpha(k + 1)) = J_2 \bar{C}_2 S(\alpha(k), \alpha(k + 1)) \]

(18)

where

\[ S(\alpha(k), \alpha(k + 1)) = \sum_{i=1}^r \sum_{j=1}^r \alpha_i(k) \alpha_j(k + 1) S_{ij} \]

\[ = \begin{bmatrix} S_0 & 0 \\
0 & S_{21} \bar{C}_2 S(\alpha(k), \alpha(k + 1)) & S_{22} \bar{C}_2 S(\alpha(k), \alpha(k + 1)) \end{bmatrix} \]

where \( S_{ij} \) are same as in (11).

From (12) and (18), we can obtain

\[ \sum_{i=1}^r \sum_{j=1}^r \alpha_i(k) \alpha_j(k + 1) \]

\[ \begin{bmatrix} Q_1 - S_{ii} - S_{ii}^T & * & * & * & * & * & * & * & * & * & * & * & * \end{bmatrix} \]

\[ \begin{bmatrix} 0 & -\gamma^2 I & * & * & * & * & * & * & * \end{bmatrix} \begin{bmatrix} \bar{A}_1 S_{ij} + \bar{B}_2 K_j \bar{C}_2 S(\alpha(k), \alpha(k + 1)) & B_{ij} - Q_I \\
0 & 0 & -\varepsilon_I \bar{M}_I^T & 0 & -\varepsilon_I I & * & * & * \end{bmatrix} \]

\[ \begin{bmatrix} \bar{N}_1 S_{ij} + N_{\varepsilon} K_j \bar{C}_2 S(\alpha(k), \alpha(k + 1)) & N_{\varepsilon} \bar{N}_{\varepsilon} \bar{M}_I^T & 0 & -\varepsilon_I I & * & * & * \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -\varepsilon_I I & * & * & * \end{bmatrix} \]

\[ \begin{bmatrix} \bar{N}_{\varepsilon} S_{ij} + N_{\varepsilon} K_j \bar{C}_2 S(\alpha(k), \alpha(k + 1)) & N_{\varepsilon} \bar{N}_{\varepsilon} \bar{M}_I^T & 0 & -\varepsilon_I I & * & * & * \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & -\varepsilon_I I & * & * \end{bmatrix} \]

which can be rewritten as follows:

\[ \begin{bmatrix} Q(\alpha(k)) - S(\alpha(k), \alpha(k + 1)) - S^{T}(\alpha(k), \alpha(k + 1)) & * & * & * & * & * & * & * & * & * & * & * & * \end{bmatrix} \]

\[ \begin{bmatrix} 0 & -\gamma^2 I & * & * & * & * & * & * \end{bmatrix} \begin{bmatrix} \bar{A}_1 S_{ij} + \bar{B}_2 K_j \bar{C}_2 S(\alpha(k), \alpha(k + 1)) & B_{ij} - Q_I \\
0 & 0 & -\varepsilon_I \bar{M}_I^T & 0 & -\varepsilon_I I & * & * & * \end{bmatrix} \]

\[ \begin{bmatrix} \bar{N}_1 S_{ij} + N_{\varepsilon} K_j \bar{C}_2 S(\alpha(k), \alpha(k + 1)) & N_{\varepsilon} \bar{N}_{\varepsilon} \bar{M}_I^T & 0 & -\varepsilon_I I & * & * & * \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -\varepsilon_I I & * & * & * \end{bmatrix} \]

\[ \begin{bmatrix} \bar{N}_{\varepsilon} S_{ij} + N_{\varepsilon} K_j \bar{C}_2 S(\alpha(k), \alpha(k + 1)) & N_{\varepsilon} \bar{N}_{\varepsilon} \bar{M}_I^T & 0 & -\varepsilon_I I & * & * & * \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & -\varepsilon_I I & * & * \end{bmatrix} \]

(19)
where $\tilde{A}(x(k)), \tilde{B}_1(x(k)), \tilde{B}_2(x(k)), \tilde{C}_1(x(k)), \tilde{C}_2, \tilde{M}, \tilde{N}_1, \tilde{N}_z$ are same as in (8) and $D_1(x(k)), D_2(x(k)), K(x(k))$ are same as in (6).

Consider the block (1, 1) of (19) and (19), we have

$$Q(x(k)) - S(x(k), x(k + 1)) - S^T(x(k), x(k + 1)) < 0$$

From $Q(x(k)) > 0$ and the above inequality, we can obtain

$$S(x(k), x(k + 1)) + S^T(x(k), x(k + 1)) > 0$$

Combining it and Lemma 6, it follows that $S(x(k), x(k + 1))$ is nonsingular. Pre- and post-multiplying (19) by $\text{diag}[S^{-T}(x(k), x(k + 1)) I I I I I I I I I I]$ and its transpose, then we have

$$
\begin{bmatrix}
S^{-T}(x(k), x(k + 1))Q(x(k))S^{-1}(x(k), x(k + 1)) & * & * & * & * & * & * & * & * & * \\
-S^{-1}(x(k), x(k + 1)) - S^{-T}(x(k), x(k + 1)) & * & * & * & * & * & * & * & * & * \\
0 & -\gamma^2 I & * & * & * & * & * & * & * & * \\
\tilde{A}(x(k)) + \tilde{B}_2(x(k))K(x(k))\tilde{C}_2 & \tilde{B}_1(x(k)) - Q(x(k + 1)) & * & * & * & * & * & * & * & * \\
\tilde{C}_1(x(k)) + D_2(x(k))K(x(k))\tilde{C}_2 & D_1(x(k)) & 0 & -I & * & * & * & * & * & * \\
0 & 0 & \varepsilon_1\tilde{M}^T & 0 & -\varepsilon_1 I & * & * & * & * & * \\
\tilde{N}_1 + N_3K(x(k))\tilde{C}_2 & N_2 & 0 & 0 & -\varepsilon_1 I - \varepsilon_1 I & * & * & * & * & * \\
0 & 0 & \varepsilon_2\tilde{M}_z^T & 0 & 0 & -\varepsilon_2 I & * & * & * & * \\
\tilde{N}_z + N_3K(x(k))\tilde{C}_2 & N_z & 0 & 0 & 0 & 0 & -\varepsilon_2 I & * & * & * \\
\end{bmatrix} < 0
$$

Because $Q(x(k)) > 0$, then it follows that

$$(S^{-1}(x(k), x(k + 1)) - Q^{-1}(x(k)))^T Q(x(k))(S^{-1}(x(k), x(k + 1)) - Q^{-1}(x(k))) \geq 0$$

which implies that

$$-Q^{-1}(x(k)) \leq S^{-T}(x(k), x(k + 1))Q(x(k))S^{-1}(x(k), x(k + 1)) - S^{-T}(x(k), x(k + 1))$$

Combining it with (20), we have

$$
\begin{bmatrix}
Q^{-1}(x(k)) & * & * & * & * & * & * & * & * & * \\
0 & -\gamma^2 I & * & * & * & * & * & * & * & * \\
\tilde{A}(x(k)) + \tilde{B}_2(x(k))K(x(k))\tilde{C}_2 & \tilde{B}_1(x(k)) - Q(x(k + 1)) & * & * & * & * & * & * & * & * \\
\tilde{C}_1(x(k)) + D_2(x(k))K(x(k))\tilde{C}_2 & D_1(x(k)) & 0 & -I & * & * & * & * & * & * \\
0 & 0 & \varepsilon_1\tilde{M}^T & 0 & -\varepsilon_1 I & * & * & * & * & * \\
\tilde{N}_1 + N_3K(x(k))\tilde{C}_2 & N_2 & 0 & 0 & 0 & -\varepsilon_1 I & * & * & * & * \\
0 & 0 & \varepsilon_2\tilde{M}_z^T & 0 & 0 & 0 & -\varepsilon_2 I & * & * & * \\
\tilde{N}_z + N_3K(x(k))\tilde{C}_2 & N_z & 0 & 0 & 0 & 0 & 0 & -\varepsilon_2 I & * & * \\
\end{bmatrix} < 0
$$

Applying the Schur complement to the above inequality, then we can obtain

$$
\begin{bmatrix}
-P(x(k)) & * & * & * \\
0 & -\gamma^2 I & * & * \\
\tilde{A}(x(k)) + \tilde{B}_2(x(k))K(x(k))\tilde{C}_2 & \tilde{B}_1(x(k)) - Q(x(k + 1)) & * & * \\
\tilde{C}_1(x(k)) + D_2(x(k))K(x(k))\tilde{C}_2 & D_1(x(k)) & 0 & -I \\
\end{bmatrix} +
\begin{bmatrix}
\frac{1}{\varepsilon_1}(\tilde{N}_1 + N_3K(x(k))\tilde{C}_2)^T(\tilde{N}_1 + N_3K(x(k))\tilde{C}_2) & * & * \\
0 & \frac{1}{\varepsilon_1}N_2^T\tilde{N}_2 & 0 \\
0 & 0 & \varepsilon_1\tilde{M}\tilde{M}^T \\
\end{bmatrix} < 0
$$

$$
\begin{bmatrix}
\frac{1}{\varepsilon_2}(\tilde{N}_z + N_3K(x(k))\tilde{C}_2)^T(\tilde{N}_z + N_3K(x(k))\tilde{C}_2) & * & * \\
0 & \frac{1}{\varepsilon_2}N_z^T\tilde{N}_2 & 0 \\
0 & 0 & \varepsilon_2\tilde{M}_z\tilde{M}_z^T \\
\end{bmatrix} < 0
$$
Applying Lemma 5 to the above inequality, it follows that

\[
\begin{bmatrix}
-P(\bar{z}(k)) & * & * & * \\
0 & -\gamma^2 I & * & * \\
\hat{A}(\bar{z}(k)) + \hat{B}_2(\bar{z}(k))K(\bar{z}(k))\hat{C}_2 & \hat{B}_1(\bar{z}(k)) & -Q(\bar{z}(k+1)) & * \\
\hat{C}_1(\bar{z}(k)) + D_2(\bar{z}(k))K(\bar{z}(k))\hat{C}_2 & D_1(\bar{z}(k)) & 0 & -I \\
\end{bmatrix} < 0
\]

\[
\begin{bmatrix}
0 & 0 & (\bar{N}_1 + N_2K(\bar{z}(k))\hat{C}_2)^T & \bar{A}_T(k)M^T(k) & 0 \\
0 & 0 & 0 & N_2^T & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} < 0
\]

\[
\begin{bmatrix}
0 & 0 & 0 & (\bar{N}_1 + N_2K(\bar{z}(k))\hat{C}_2)^T & \bar{A}_T(k)M^T(k) \\
0 & 0 & 0 & 0 & N_2^T \\
0 & 0 & 0 & 0 & 0 \\
M_2A_z(k)\bar{N}_1 + N_3K(\bar{z}(k))\hat{C}_2 & M_2A_z(k)N_2 & 0 & 0 \\
\end{bmatrix} < 0
\]

i.e.,

\[
\begin{bmatrix}
-P(\bar{z}(k)) & * & * & * \\
0 & -\gamma^2 I & * & * \\
A & B & -Q(\bar{z}(k+1)) & * \\
C & D & 0 & -I \\
\end{bmatrix} < 0
\]

where \(A, B, C, D\) are same as in (16).

Applying the Schur complement to the above inequality, then we have

\[
\begin{bmatrix}
A^TP(\bar{z}(k)+1)A - P(\bar{z}(k)) + C^TC & A^TP(\bar{z}(k)+1)B + C^TD \\
B^TP(\bar{z}(k)+1)A + D^TC & B^TP(\bar{z}(k)+1)B + D^TD - \gamma^2 I \\
\end{bmatrix} < 0
\]

which implies (17) is less than zero for nonzero vector \([\bar{x}^T(k) w^T(k)]^T\), then

\[
J_\infty = \sum_{i=0}^{\infty} (\bar{z}^T(k)z(k) - \gamma^2 w^T(k)w(k)) < V(0) - V(\infty) \leq \bar{x}^T(0)P(\bar{z}(0))\bar{x}(0)
\]

Under zero initial condition, \(J_\infty < 0\), which implies that the transformed closed-loop uncertain fuzzy system (7) satisfies \(H_\infty\) performance bound \(\gamma\). Moreover, because (17) is less than zero, we can obtain \(V(k+1) - V(k) < 0\) for \(w(k) = 0\), that means that the transformed closed-loop uncertain fuzzy system (7) is asymptotically stable. Then system (5) satisfies \(H_\infty\) performance bound \(\gamma\) and is asymptotically stable. Thus, the proof is complete. \(\square\)

**Remark 8.** (i) Theorem 7 provides a sufficient condition for designing SOF \(H_\infty\) controllers in terms of solutions to a set of parameterized linear matrix inequalities (PLMIs). Since \(z(k)\) appears in the PLMIs, the inequalities are nonlinear in \(z(k)\). It will be very difficult to directly solve the PLMIs involved in Theorem 7 because the PLMIs need to be checked for all values of \(z(k)\), which results in solving an infinite number of LMIs. There have been many approaches [5,13,17,21,25,26,28] for converting the PLMIs into a finite number of LMIs, in particular, the method in [21] is a sufficient and necessary condition, but the computational burden could be very heavy. The method in [28] is with much fewer variables involved and much more efficient computation than other approaches. Inhere, we apply the technique given in [28] to fulfill the task. As a result, an LMI-based condition is obtained in Theorem 9.

(ii) In contrast to the existing approaches with the constraints on Lyapunov matrices \(P\) (i.e., \(P^{-1} = \begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} \end{bmatrix}\) in [18]; \(C_2)\) \(P^{-1} = MC_{2i}, 1 \leq i \leq r\) in [2,12]), in this paper, a lower triangular structure constraint is only imposed on the introduced parameter-dependent slack variable \(S(\bar{z}(k), \bar{z}(k+1)) = \begin{bmatrix} S_0 & 0 \\ S_1(\bar{z}(k), \bar{z}(k+1)) & S_2(\bar{z}(k), \bar{z}(k+1)) \end{bmatrix}\) for designing SOF output feedback controllers, and the Lyapunov matrix \(P(\bar{z}(k))\) is with no structure constraint. The new feature is
very helpful for giving less conservative results for SOF control synthesis. The fact will be shown by Example 14 in Section 4.

(iii) Notice that the introduced slack variable \( S(\bar{x}(k), \bar{x}(k + 1)) = \begin{bmatrix} S_0 & 0 \\ S_{21}(\bar{x}(k), \bar{x}(k + 1)) & S_{22}(\bar{x}(k), \bar{x}(k + 1)) \end{bmatrix} \) is parameter-dependent in Theorem 7. However, we propose a method for designing SOF \( H_\infty \) controllers for fuzzy control systems in [4] by introducing a parameter-independent slack variable. In fact, if \( S = \begin{bmatrix} S_0 & 0 \\ S_{21} & S_{22} \end{bmatrix} \) in Theorem 7, then the condition in Theorem 7 will reduce to the condition in [4]. The role of the parameter-dependent variable \( S(\bar{x}(k), \bar{x}(k + 1)) \) will be illustrated in Example 14 in Section 4, see Table 2.

**Theorem 9.** Given a prescribed \( H_\infty \) performance index \( \gamma > 0 \), if there exist a symmetric matrix positive \( Q_i > 0 \), and matrices \( S_{ij}, L_i, 1 \leq i, l \leq r \), with (11) satisfying the following LMIs:

\[
A_{ii} < 0, \quad 1 \leq i, \ l \leq r \\
\frac{1}{r - 1} A_{ii} + \frac{1}{2} (A_{ij} + A_{ji}) < 0, \quad 1 \leq i \neq j, \ 1 \leq l \leq r
\]

where \( A_{ij} \) are same as in (13). Then the closed-loop system (5) with the fuzzy controller (4) with (14) is asymptotically stable and with an \( H_\infty \) performance bound \( \gamma \).

**Proof.** Based on Theorem 7 and using the technique in [28], the proof is easily obtained and omitted. \( \square \)

**Remark 10.** Theorem 9 gives an LMI-based condition for designing SOF \( H_\infty \) controllers, which can be solved efficiently via LMI Control Toolbox. In Theorem 11, it is shown that Theorem 9 (the new proposed method) can give less conservative design than Lemma 3 (the existing method in [18]).

**Theorem 11.** If the condition of Lemma 3 holds, then the condition of Theorem 9 also holds.

**Proof.** If there exist a symmetric matrix \( X = \begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} \end{bmatrix} \), and matrices \( L_i = [L_{i0} \ 0], 1 \leq i \leq r \), satisfying (9), then choose

\[
Q_i = X = \begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} \end{bmatrix}, \quad S_{ii} = X = \begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} \end{bmatrix}, \quad 1 \leq i, \ l \leq r
\]

therefore, \( A_{ij} \) in (21) is equal to \( M_{ij} \) in (9). Then (9) holds implies that (21) holds, i.e., the condition of Theorem 9 holds. Thus, the proof is complete. \( \square \)

**Remark 12.** Theorem 11 shows that the result given by Theorem 9 is less conservative than that given in [18] (Lemma 3). The numerical comparisons between Theorem 9 and Lemma 3 are given in Example 13. For comparing with the techniques in [2,12], a numerical example (Example 14 in Section 4) is given for showing that the new proposed method can provide less conservative results than the ones in [2,12] (the methods CMOb1, CMOb2, CMOb3).

4. Example

In this section, all numerical tests have been performed in a PC with Pentium 4 3.1 GHz, 1024 MB RAM, using the LMI Toolbox in Matlab 7.0. Two examples will be given for illustrating the effectiveness of the proposed method. In Example 13, a practical truck-trailer model is considered and a SOF controller for backing up a truck-trailer is designed, and the corresponding simulink results will be given to validate the conclusion of Theorem 11.

Besides the technique in Theorem 7, there are many existing ones, which have been developed for designing SOF controllers, such as the ones in [2,12,18]. These techniques all can give the corresponding sufficient condition by PLMs, such as the ones in [5,13,28]. On the other hand, there are many methods to turn PLMs into a finite number of LMIs. The combinations of two class of techniques can give different LMI-based conditions for designing SOF controllers, which all will be applied to Example 14 to show the effectiveness of the new technique in this paper.
Example 13. A truck-trailer model taken from [18], is given as follows:

\[ x_1(k + 1) = \left(1 - \frac{vt}{L}\right)x_1(k) + \frac{vt}{l}u(k) \]

\[ x_2(k + 1) = \frac{vt}{L}x_1(k) + x_2(k) + 0.2w(k) \]

\[ x_3(k + 1) = x_3(k) + vt\sin(\theta_k) + 0.1w(k) \]

\[ z(k) = C_1x(k) + \frac{vt}{l}u(k) \]

\[ y(k) = C_2x(k) \]

where \( \theta_k = x_2(k) + (vt/2L)x_1(k) \). The nonlinear system can be modeled as a two rules fuzzy model.

Model Rule 1: If \( \theta_k \) is \( z_1 \) then \( x(k + 1) = (A_1 + M_1N_1)x(k) + B_1w(k) + B_2u(k) \)

Model Rule 2: If \( \theta_k \) is \( z_2 \) then \( x(k + 1) = (A_2 + M_1N_1)x(k) + B_1w(k) + B_2u(k) \)

where

\[
A_i = \begin{bmatrix}
1 - \frac{vt}{L} & 0 & 0 \\
\frac{vt}{L} & 1 & 0 \\
\frac{(vt)^2}{2L}d_i & vt d_i & 1
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
0 \\
0.2 \\
0.1
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
\frac{vt}{l} \\
0 \\
0
\end{bmatrix}
\]

\[
M_1 = \begin{bmatrix}
0 \\
0 \\
0.0023
\end{bmatrix}, \quad N_1 = \begin{bmatrix}
\frac{vt}{L} \\
2L \\
1 \\
0
\end{bmatrix}^T
\]

\( d_1 = 1, d_2 = 0.01/\pi \)

The system parameters are

\( l = 2.8, \quad L = 5.5, \quad v = -1, \quad C_1 = [0, 0, 0.001], \quad C_2 = [7, -2, 0.03] \)

and \( \|A^T A\| \leq 1 \). Fuzzy sets \( z_i, i = 1, 2, \) are for \( \theta_k \), respectively, around \( \theta_k = 0 \), and \( \pm 179.997\pi/180 \).

\( z_1 = \frac{h_1}{h_1 + h_2}, \quad z_2 = \frac{h_2}{h_1 + h_2} \)

where

\( h_1 = e^{-5\theta_k^2}, \quad h_2 = \max\{e^{-5(\theta_k-\pi)^2}, e^{-5(\theta_k+\pi)^2}\} \)

Using formula (3), the obtain transformation matrix is

\[
T = \begin{bmatrix}
0.1321 & 0.2747 & -0.0041 \\
-0.0377 & 0.9615 & 0.0006 \\
0.0006 & 0.0006 & 1.0000
\end{bmatrix}
\]

In order to validate the conclusion in Theorem 11, Lemma 3 and Theorem 9 is applicable for the example. The solver \textit{mincx} in LMI tool box is used to minimized \( \gamma \) (the solver can automatically search the minimization of \( \gamma \) under LMI constraints). By using Lemma 3, the following computational results are obtained:

\[
X = \begin{bmatrix}
0.9427 & 0 & 0 \\
0 & 0.0907 & 1.7675 \\
0 & 1.7675 & 117.1654
\end{bmatrix}, \quad L_{11} = 0.0882, \quad L_{21} = 0.0882 
\]

\( K_1 = 0.0935, \quad K_2 = 0.0935 \)

\( \gamma_{opt} = 4.3712 \)
By using Theorem 9, the following computational results are obtained:

\[
Q_1 = \begin{bmatrix}
1.0297 & 0.0929 & 0.1881 \\
0.0929 & 0.1385 & 1.7008 \\
0.1881 & 1.7008 & 113.7170
\end{bmatrix},
Q_2 = \begin{bmatrix}
0.8247 & 0.1071 & 0.0308 \\
0.1071 & 0.1488 & 1.6981 \\
0.0308 & 1.6981 & 110.3349
\end{bmatrix}
\]

\[
S_{11} = \begin{bmatrix}
0.7939 & 0 & 0 \\
0.0846 & 0.1483 & 1.9658 \\
-0.0150 & 1.9654 & 120.8601
\end{bmatrix},
S_{12} = \begin{bmatrix}
0.7939 & 0 & 0 \\
0.0857 & 0.1546 & 1.9178 \\
0.0772 & 1.9096 & 116.5412
\end{bmatrix}
\]

\[
S_{21} = \begin{bmatrix}
0.7939 & 0 & 0 \\
0.1057 & 0.1488 & 1.7478 \\
-0.1508 & 1.6969 & 114.1850
\end{bmatrix},
S_{22} = \begin{bmatrix}
0.7939 & 0 & 0 \\
0.1049 & 0.1553 & 1.6983 \\
0.0470 & 1.6485 & 110.3337
\end{bmatrix}
\]

\[L_{11} = 0.2869, \quad L_{21} = 0.3860\]

\[K_1 = 0.3589, \quad K_2 = 0.4829\]

\[\gamma_{\text{opt}} = 2.4747\]

From the above computational results, it can be seen that the new approach (Theorem 9) gives smaller \(H_\infty\) performance, which illustrates the effectiveness of the new method and also validates the conclusion of Theorem 11, i.e., the new method (Theorem 9) can give less or at least the same conservative results than Lemma 3 (an existing one), see Remark 10.

Moreover, a simulation will also be given to illustrate the conclusion of Theorem 11. The simulation results with the controllers designed by using Lemma 3 and Theorem 9, respectively, are given in Figs. 1–4 with an initial state is \(x(0) = [12 - 5]\), and \(w(k) = \sin(k)/(k + 1)\).

From Figs. 1 to 4, it can been seen the closed-loop system with the designed controller is asymptotically stable, and the \(H_\infty\) performance is guaranteed.

**Example 14.** Consider the following T–S fuzzy model with three fuzzy rules:

\[
x(k + 1) = \sum_{i=1}^{3} z_i(k) (A_i + M_1 \Delta N_1)x(k) + \sum_{i=1}^{3} z_i(k) B_{1i} w(k) + \sum_{i=1}^{3} z_i(k) B_{2i} u(k)
\]

\[
z(k) = \sum_{i=1}^{3} z_i(k) C_{1i} x(k) + \sum_{i=1}^{3} z_i(k) D_{1i} w(k) + \sum_{i=1}^{3} z_i(k) D_{2i} u(k)
\]

\[y(k) = C_2 x(k)\]
where

\[ A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0.8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.3 & -2 \\ 1 & 0.3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1.1 & -3 \\ 1 & 0.6 \end{bmatrix} \]

\[ B_{11} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}, \quad B_{13} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \]

\[ B_{21} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_{23} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

\[ C_{11} = [0 \ 0.3], \quad C_{12} = [0 \ 0.1], \quad C_{13} = [0 \ 0.1] \]

\[ D_{11} = D_{12} = D_{13} = 0.1, \quad D_{21} = 3, \quad D_{22} = D_{23} = 1 \]

\[ C_2 = [1 \ 0], \quad M_1 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \quad N_1 = [0 \ 1] \]

\[ \| \Delta^T \Delta \| \leq 1 \] and the membership functions \( z_1, z_2 \) and \( z_3 \) are depicted in Fig. 5.
Because the proposed condition in Theorem 7 is based on PLMIs, those techniques [5,13,17,21,25,26,28], which turn the PLMIs into a finite number of LMIs, are all applicable to Theorem 7 for different SOF controller design conditions. What it follows, the given condition in Theorem 7 (i.e., the proposed PLMI condition) and the techniques of in [5,13,28] (i.e., these approaches turning PLMI into LMIs) will be combined for designing SOF controllers. These obtained conditions are given as follows:

- The condition given by the combination of Theorem 7 and the technique in [28] is denoted as CMN1 (i.e., Theorem 9).
- The condition given by the combination of Theorem 7 and the technique in [13] is denoted as CMN2.
- The condition given by the combination of Theorem 7 and the technique in [5] is denoted as CMN3.

Apply the above conditions (CMN1, CMN2, CMN3) to the example, and using the solver mincx in LMI tool box to minimized $\gamma$, then the obtained $H_\infty$ performance and corresponding cputimes are shown in Table 1.

From Table 1, it can be seen that, the conditions in [5,13] can improve the results in Theorem 9 (i.e., CMN1). However, more cputimes are expended, see Remark 8(i).
Table 2
Combinations of Theorem 7 with $S_{il} = S_{11}$ and the PLMI techniques in [5,13,28].

<table>
<thead>
<tr>
<th>Technique</th>
<th>$\gamma$ (cputime)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CMNa1</td>
<td>2.4381 (010.7969 s)</td>
</tr>
<tr>
<td>CMNa2</td>
<td>2.2389 (119.2813 s)</td>
</tr>
<tr>
<td>CMNa3</td>
<td>2.1111 ($2.8112 \times 10^3$ s)</td>
</tr>
</tbody>
</table>

Combinations of the technique in [4] and the PLMI techniques in [5,13,28].

Table 3
Combinations of the technique in [18] and the PLMI techniques in [5,13,28].

<table>
<thead>
<tr>
<th>Technique</th>
<th>$\gamma$ (cputime)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CMOa1</td>
<td>4.6237 (4.4219 s)</td>
</tr>
<tr>
<td>CMOa2</td>
<td>4.0611 (10.2188 s)</td>
</tr>
<tr>
<td>CMOa3</td>
<td>3.8021 (262.4063 s)</td>
</tr>
</tbody>
</table>

Note that the slack variable $S(z(k), z(k + 1))$ is parameter-dependent, and the condition of Theorem 7 with $S_{il} = S_{11} = S_{211} = S_{221}$, $1 \leq i, l \leq r$ (which implies that the introduced slack variable is parameter-independent) has been presented in [4]. In order to illustrate the effectiveness of the parameter-dependent variable, the condition of Theorem 7 with $S_{il} = S_{11} = S_{211} = S_{221}$, $1 \leq i, l \leq r$ will be applied to the example, and these obtained conditions are given as follows:

- The condition given by the combination of Theorem 7 with $S_{il} = S_{11} = S_{211} = S_{221}$, $1 \leq i, l \leq r$ and the technique in [28] is denoted as CMNa1.
- The condition given by the combination of Theorem 7 with $S_{il} = S_{11} = S_{211} = S_{221}$, $1 \leq i, l \leq r$ and the technique in [13] is denoted as CMNa2.
- The condition given by the combination of Theorem 7 with $S_{il} = S_{11} = S_{211} = S_{221}$, $1 \leq i, l \leq r$ and the technique in [5] is denoted as CMNa3.

Applying CMNa1, CMNa2, CMNa3 to the example, the obtained results are shown in Table 2.

From Tables 1 and 2, it can be seen that the parameter-dependent slack variable $S(z(k), z(k + 1))$ can reduce the conservatism.

Moreover, for SOF controllers design, some approaches have been developed in [18] (diagonal structure constraint about Lyapunov matrix, i.e., $P^{-1} = \begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} \end{bmatrix}$) and [2,12] (inserting an equality condition constraint on Lyapunov matrix, i.e., $C_{2i} P^{-1} = M C_{2i}$, $1 \leq i \leq r$). In order to compare the new proposed method (PLMIs condition) in Theorem 7 and the existing ones in [2,12,18], the existing ones (PLMIs conditions) will, respectively, be combined with the conditions in [5,13,28] (turning PLMIs into LMIs). These obtained conditions are given as follows:

- The condition given by the combination of the technique in [18] and the technique in [28] is denoted as CMOa1 (i.e., Lemma 3).
- The condition given by the combination of the technique in [18] and the technique in [13] is denoted as CMOa2.
- The condition given by the combination of the technique in [18] and the technique in [5] is denoted as CMOa3.
- The condition given by the combination of the technique in [2,12] and the technique in [28] is denoted as CMOb1.
- The condition given by the combination of the technique in [2,12] and the technique in [13] is denoted as CMOb2.
- The condition given by the combination of the technique in [2,12] and the technique in [5] is denoted as CMOb3.

These obtained conditions (CMOa1, CMOa2, CMOa3, CMOb1, CMOb2, CMOb3) are also applied to the example for further comparisons. The optimal $H_{\infty}$ performance bounds and the corresponding cputimes are given in Tables 3 and 4.
Table 4
Combinations of the technique in [2,12] and the PLMI techniques in [5,13,28].

<table>
<thead>
<tr>
<th>Technique</th>
<th>( \gamma ) (cputime)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CMOb1</td>
<td>4.5304 (8.0313 s)</td>
</tr>
<tr>
<td>CMOb2</td>
<td>3.9970 (12.2500 s)</td>
</tr>
<tr>
<td>CMOb3</td>
<td>3.7494 (278.2969 s)</td>
</tr>
</tbody>
</table>

From Tables 1 to 4, it can be seen that the new method (Theorem 7) can give less conservative results than existing ones (diagonal structure constraint on Lyapunov matrix [18] or inserting an equality condition constraint about Lyapunov matrix [2,12]), when they are combined with other techniques (turning PLMIs into LMIs) in [5,13,28], see Remark 8(ii).

5. Conclusion

In this paper, the problem of designing \( H_{\infty} \) SOF controllers for T–S fuzzy systems has been studied. By considering the properties of system output matrices, a new method for designing \( H_{\infty} \) SOF controllers is given. The comparisons with the existing methods in the literature have been done, which show that the new proposed design method can provide less conservative designs than the existing results. Numerical examples have shown the effectiveness of the proposed design method.

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References


