**H∞ control design for fuzzy discrete-time singularly perturbed systems via slow state variables feedback: An LMI-based approach**

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This paper addresses the $H_\infty$ control problem via slow state variables feedback for discrete-time fuzzy singularly perturbed systems. At first, a method of evaluating the upper bound of singular perturbation parameter $\epsilon$ with meeting a prescribed $H_\infty$ performance bound requirement is given. Subsequently, two methods for designing $H_\infty$ controllers via slow state variables feedback are presented in terms of solutions to a set of linear matrix inequalities (LMIs). In particular, one of them can be used to improve the upper bound of the singular perturbation parameter $\epsilon$. Finally, two numerical examples are given to illustrate the effectiveness of the proposed methods.

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**1. Introduction**

Slow and fast dynamic phenomena in control systems often occur due to the presence of small “parasitic” parameters, such as motor control systems, electronic circuits, magnetic-ball suspension systems, and so on. In a state space framework, such systems are commonly modeled by a mathematical description of singular perturbations, where a small parameter $\epsilon$ is exploited to determine the degree of separation between slow and fast parts of the dynamical system. Due to the very small singular perturbation parameter $\epsilon$, the analysis and synthesis approaches for normal systems often lead to ill-conditioned results. Therefore, a so-called reduction technique with a two-step design methodology [17] is widely adopted for overcoming the difficulty. Firstly, through the separate stabilization of two lower dimensional subsystems in two different time scales, a composite stabilizing controller is synthesized from separate stabilizing controllers of the two subsystems, where the controller could be determined without the knowledge of the small singular perturbation parameter. In the past several decades, many control problems of singularly perturbed systems have attracted considerable attentions, see the survey paper [23] and the references therein.

In control theory, a well-known convex optimization technique, i.e., linear matrix inequality (LMI) technique, has been extensively exploited to solve control problems [3]. In contrast to Riccati approaches, linear matrix inequalities (LMIs) can be formulated as convex optimization problems that are amenable to compute solution and can be solved effectively [3]. Another good feature of LMIs is their ability of adding constraints to the parametrical optimization problem provided...
they are themselves linear with respect to unknowns [12]. In particular, for $H_\infty$ synthesis, it has the merit of eliminating the regularity restrictions attached to the Riccati-based solutions [11]. Motivated by the merits of the LMI formulations, some LMI-based controller design approaches for singularly perturbed systems have been developed in [7–10,28] recently.

On the other hand, there has been a great deal of interest in using Takagi–Sugeno (T–S) fuzzy models to approximate nonlinear systems, and many control problems of nonlinear systems have been widely studied based on T–S fuzzy systems, see [14,15,26,29,30] and the references therein. In particular, the controller design conditions for the state feedback [19], static output feedback [21], and dynamic output feedback [2] cases are exploited in terms of solutions of LMIs for fuzzy singularly perturbed systems. Moreover, the methods of designing $H_\infty$ state feedback controllers with pole placement constraints are given in [1]. In most cases, the “fast” dynamics of singularly perturbed systems are not adequately modeled and are, therefore, neglected in order to simplify the design [24,6]. In practice the fast variables are not directly measurable sometimes, for example, the flexible variables (modeled as fast variables) of the flexible link manipulators [22]. Therefore, the study on the problem of designing state feedback controllers by only using slow state variables of singularly perturbed systems is necessary.

Because most of the existing synthesis techniques for singularly perturbed systems are independent of $\epsilon$ for avoiding to obtain ill-conditioned results, it is of great importance to find the bound of $\epsilon$ for ensuring the stability of the closed-loop systems. As a result, it has attracted increasing interest in the past several decades. In [5], an approach to characterize and compute the stability bound is presented for continuous-time singularly perturbed systems. By considering critical stability criteria with a bialternate product, systematic approaches to determine the exact stability bound of discrete-time singularly perturbed systems are given in [13,18]. Moreover, an algorithm for finding the upper bound of the singular perturbation parameter for $D$-stability is presented in [16]. However, by the authors’ knowledge, the topic of evaluating the upper bounds of singular perturbation parameters for nonlinear discrete-time singularly perturbed systems with meeting $H_\infty$ performance requirements has not been studied. In this paper, the topic will be addressed by using the two lemmas that are proposed for linear singularly perturbed systems in [8], which are given in Appendices A and B.

In this paper, a method of evaluating the upper bound of the singular perturbation parameter $\epsilon$ for discrete-time fuzzy singularly perturbed systems with meeting a prescribed $H_\infty$ performance bound requirement is given. Furthermore, two $H_\infty$ controller design methods via slow state variables feedback are presented in terms of solutions to a set of LMIs. In contrast to the conventional design methods [23], the new design methods are with twofold advantages. One is that the two design methods are based on LMIs, which can eliminate the regularity restrictions attached to the Riccati-based solution. The other is that one of the two methods can be used to improve the upper bound of singularly perturbed parameter $\epsilon$ at the stage of control design, which implies that the tradeoff between the $H_\infty$ performance index and the upper bound of the singular perturbation parameter $\epsilon$ is considered in the design. Thus, the new controller design method can overcome the disadvantage that the allowable upper bound of the singular perturbation parameter of the closed-loop system with the controllers designed by the existing ones is too small to be used. This paper is organized as follows. Section 2 presents system description and some preliminaries. In Section 3, a sufficient condition is derived for evaluating the upper bound $\epsilon^\ast$ of $\epsilon$ subject to a prescribed $H_\infty$ performance constraint. Moreover, two new LMI-based $H_\infty$ controller design methods are presented. In particular, one of them can improve the upper bound of the singular perturbation parameter $\epsilon$ by designing controllers. The validity of these approaches is illustrated by two numerical examples in Section 4. Finally, Section 5 concludes the paper.

**Notation:** $\mathbb{R}^n$ denotes the set which consists of real $n$-vectors ($n \times 1$ matrices). For a symmetric block matrix, $(*)$ is used for the blocks induced by symmetry, for example,

\[
\begin{bmatrix}
M_{11} & * & * \\
M_{21} & M_{22} & * \\
M_{31} & M_{32} & M_{33}
\end{bmatrix} =
\begin{bmatrix}
M_{11} & M_{12}^T & M_{13}^T \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{bmatrix}
\]

The superscript $T$ stands for matrix transposition and the notation $M^{-T}$ denotes the transpose of the inverse matrix of $M$.

**2. System description and some preliminaries**

A class of nonlinear singularly perturbed systems under consideration are described by the following fuzzy system model:

**Plant Rule i :**

IF $\nu_1(k)$ is $M_{i1}$ and $\nu_2(k)$ is $M_{i2}$, ..., $\nu_p(k)$ is $M_{ip}$, THEN

\[
\begin{align*}
x_1(k+1) &= A_{i1}^T x_1(k) + \epsilon A_{i2}^T x_2(k) + B_{w1}^T w(k) + B_{u1}^T u(k) \\
x_2(k+1) &= A_{i1}^T x_1(k) + \epsilon A_{i2}^T x_2(k) + B_{w2}^T w(k) + B_{u2}^T u(k) \\
z(k) &= C_{i1}^T x_1(k) + \epsilon C_{i2}^T x_2(k) + D_{uw}^T w(k) + D_{uu}^T u(k)
\end{align*}
\]

where $i = 1, 2, \ldots, r$, $r$ is the number of IF–THEN rules. $M_{ij}$, $1 \leq i \leq r$, $1 \leq j \leq p$ are fuzzy sets. $\nu_i(k)$ are the premise variables, $x_1(k) \in \mathbb{R}^n$ and $x_2(k) \in \mathbb{R}^n$ are respectively the slow and fast state vectors, $u(k) \in \mathbb{R}^n$ is the control input, $w(k) \in \mathbb{R}^m$ is the disturbance, $z(k) \in \mathbb{R}^l$ is the controlled output, the matrices $A_{i1}$, $A_{i2}$, $B_{w1}^T$, $B_{u1}^T$, $B_{w2}^T$, $B_{u2}^T$, $C_{i1}$, $C_{i2}$, $D_{uw}^T$ and $D_{uu}^T$ are of appropriate dimensions. $\epsilon > 0$ is a singular perturbation parameter, which determines the degree of separation between the “slow” and “fast” modes of the system [1].
Denote
\[ \sigma_i(v(k)) = \prod_{j=1}^{p} \mu_{ij}(v_j(k)) \]

\( \mu_{ij}(v_j(k)) \) is the grade of membership of \( v_j(k) \) in \( M_{ij} \), where it is assumed that
\[ \sum_{i=1}^{r} \sigma_i(v(k)) > 0, \quad \sigma_i(v(k)) \geq 0 \quad i = 1, 2, \ldots, r \]

Let \( \alpha_i(v(k)) = \frac{\sigma_i(v(k))}{\sum_{i=1}^{r} \sigma_i(v(k))} \), then
\[ 0 \leq \alpha_i(v(k)) \leq 1, \quad \text{and} \quad \sum_{i=1}^{r} \alpha_i(v(k)) = 1 \] (2)

\( \alpha_i(v(k)), i = 1, \ldots, r \) are said to be normalized membership functions. Then, the T–S fuzzy model of (1) is inferred as follows:
\[ x_1(k + 1) = \sum_{i=1}^{r} \alpha_i(v(k)) \left( A_{i1} x_1(k) + \epsilon A_{i12} x_2(k) + B_{w1} w(k) + B_{u1} u(k) \right) \]
\[ x_2(k + 1) = \sum_{i=1}^{r} \alpha_i(v(k)) \left( A_{i2} x_1(k) + \epsilon A_{i22} x_2(k) + B_{w2} w(k) + B_{u2} u(k) \right) \] (3)
\[ z(k) = \sum_{i=1}^{r} \alpha_i(v(k)) \left( C_{i1} x_1(k) + \epsilon C_{i2} x_2(k) + D_{w1} w(k) + D_{u1} u(k) \right) \]

which can be rewritten as follows:
\[ x(k + 1) = A(x(k)) E x(k) + B_w(x(k)) w(k) + B_u(x(k)) u(k) \]
\[ z(k) = C_x(x(k)) E x(k) + D_{w1}(x(k)) w(k) + D_{u1}(x(k)) u(k) \] (4)

where
\[ A(x(k)) = \sum_{i=1}^{r} \alpha_i(v(k)) A_i, \quad B_w(x(k)) = \sum_{i=1}^{r} \alpha_i(v(k)) B_{wi}, \]
\[ B_u(x(k)) = \sum_{i=1}^{r} \alpha_i(v(k)) B_{ui}, \quad C_x(x(k)) = \sum_{i=1}^{r} \alpha_i(v(k)) C_i, \]
\[ D_{w1}(x(k)) = \sum_{i=1}^{r} \alpha_i(v(k)) D_{wi}, \quad D_{w2}(x(k)) = \sum_{i=1}^{r} \alpha_i(v(k)) D_{w2}, \]
\[ E = \begin{bmatrix} I_{n_x \times n_x} & 0 \\ 0 & \epsilon I_{n_z \times n_z} \end{bmatrix} \]
\[ A_i = \begin{bmatrix} A_{i11} & A_{i12} \\ A_{i21} & A_{i22} \end{bmatrix}, \quad B_{wi} = \begin{bmatrix} B_{wi1} \\ B_{wi2} \end{bmatrix}, \quad \epsilon \]
\[ B_{ui} = \begin{bmatrix} B_{ui1} \\ B_{ui2} \end{bmatrix}, \quad C_i = \begin{bmatrix} C_{i1} \\ C_{i2} \end{bmatrix} \] (5)

In this paper, the concept of parallel distributed compensation (PDC) is used to design fuzzy controllers, i.e., the designed fuzzy controller shares the same fuzzy sets with the fuzzy model in the premise parts. The more details can be found in [25]. For the fuzzy model (1), the following slow state feedback controller is adopted, where \( K^i_{ui}, 1 \leq i \leq r \) are the parameters to be designed.

Control Rule i:
IF \( v_i(k) \) is \( M_{i1} \) and \( v_2(k) \) is \( M_{i2}, \ldots, v_p(k) \) is \( M_{ip} \)
THEN \( u(k) = K^i_{ui} x_1(k) \) (6)

Because the control rules are the same as the plant rules, the fuzzy controller can be obtained as follows:
\[ u(k) = \sum_{i=1}^{r} \alpha_i(v(k)) K^i_{ui} x_1(k) = \sum_{i=1}^{r} \alpha_i(v(k)) K^i \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \] (7)

where
\[ K^i = \begin{bmatrix} K^i_{a} & 0 \end{bmatrix} \] (8)
Combining (7) and (4), the resulting closed-loop system is given as follows:

\[ x(k + 1) = (A(x(k))z_\text{ex} + B_u(x(k))K(x(k)))x(k) + B_w(x(k))w(k) \]
\[ z(k) = (C_z(x(k))E_z + D_z(x(k))K(x(k)))x(k) + D_w(x(k))w(k) \]  

(9)

where

\[ K(x(k)) = \sum_{i=1}^{r} \alpha_i(v(k)) \begin{bmatrix} K^*_0 & 0 \end{bmatrix} \]

(10)

The replacement of \( x_{\text{ex}} \) by \( \xi \) in system (9) will result in the equivalent system

\[ \xi(k + 1) = E_r(A(x(k)) + B_u(x(k))K(x(k)))\xi(k) + E_rB_w(x(k))w(k) \]
\[ z(k) = (C_z(x(k)) + D_z(x(k))K(x(k)))\xi(k) + D_w(x(k))w(k) \]  

(11)

where

\[ \xi(k) = \begin{bmatrix} x_1(k) \\ \xi_2(k) \end{bmatrix} \]

The \( H_\infty \) norm in [20] for nonlinear discrete-time systems is applicable for nonlinear discrete-time singularly perturbed systems. The definition is given as follows:

**Definition 1** [20]. Given a real number \( \gamma > 0 \), it is said that the \( H_\infty \) norm of the closed-loop system (11) is less than or equal to \( \gamma \) (i.e., the exogenous signals are locally attenuated by \( \gamma \)) if there exists a neighborhood \( U \) of \( x = 0 \) such that for every positive integer \( N \) and for every \( w \in L_2(0,N,R^{n_w}) \) for which the state trajectory of the closed-loop system (11) starting \( x(0) = 0 \) remains in \( U \) for all \( k \in [0,N], \) the response \( z \in L_2(0,N,R^{n_z}) \) of (11) satisfies

\[ \sum_{i=0}^{N} \|z_k\|^2 \leq \gamma^2 \sum_{i=0}^{N} \|w_k\|^2, \text{ for all } N \]

In this paper, the following problems will be addressed.

### 2.1. Evaluation of the upper bound of \( \epsilon \) with meeting stability and \( H_\infty \) performance bound requirement

Let \( \gamma > 0 \) be a given constant and the gains \( K^* \) be given. Find an \( \epsilon^* > 0 \) as big as possible such that the system (1) with (7) is asymptotically stable and its \( H_\infty \)-norm is less than or equal to \( \gamma \) for any \( \epsilon \in (0,\epsilon^*]. \)

### 2.2. \( H_\infty \) controller designs without the consideration of improving the upper bound of \( \epsilon \)

Let \( \gamma > 0 \) be a given constant, find gains \( K^*_i \) \( i = 1, \ldots, r, \) and there exists a positive scalar \( \epsilon, \) such that the system (1) with (7) is asymptotically stable and its \( H_\infty \)-norm is less than or equal to \( \gamma \) for any \( \epsilon \in (0,\epsilon^*]. \)

### 2.3. \( H_\infty \) controller design with the consideration of improving the upper bound of \( \epsilon \)

Let \( \gamma > 0 \) be a given constant and \( \epsilon > 0 \) be a prescribed upper bound of the singular perturbation parameter \( \epsilon. \) Find gains \( K^*_i \) \( i = 1, \ldots, r, \) and an \( \epsilon^* > 0 \) such that the system (1) with (7) is asymptotically stable and its \( H_\infty \)-norm is less than or equal to \( \gamma \) for any \( \epsilon \in (0,\epsilon^*]. \)

The following lemmas will be used in this sequel.

**Lemma 2.** If there exists a symmetric positive-definite matrix \( P(x(k)) \) such that the following LMIs hold,

\[
\begin{bmatrix}
\Phi_{11}(k, k + 1) & * \\
\Phi_{21}(k, k + 1) & \Phi_{22}(k, k + 1)
\end{bmatrix} < 0, \quad \text{for } \epsilon \in (0,\epsilon^*]  
\]

(12)

where \( A(x(k)), B_u(x(k)), B_w(x(k)), C_z(x(k)), D_z(x(k)) \) and \( D_w(x(k)) \) are the same as in (5), \( K(x(k)) \) is the same as in (10), and

\[
\Phi_{11}(k, k + 1) = -P(x(k)) + (A(x(k)) + B_u(x(k))K(x(k)))^T \\
\times E_rP(x(k + 1))E_r(A(x(k)) + B_u(x(k))K(x(k))) \\
+ \frac{1}{j}(C_z(x(k)) + D_z(x(k))K(x(k)))^T \\
\times (C_z(x(k)) + D_z(x(k))K(x(k))) \\
\Phi_{21}(k, k + 1) = B_u^TP(x(k))E_rP(x(k + 1))E_r(A(x(k)) + B_u(x(k))K(x(k))) \\
+ \frac{1}{j}D_z^T(x(k))(C_z(x(k)) + D_z(x(k))K(x(k))) \]

(13)

\[
\Phi_{22}(k, k + 1) = B_u^TP(x(k))E_rP(x(k + 1))E_rB_w(x(k)) + \frac{1}{j}D_z^T(x(k))D_z(x(k)) - \gamma I
\]
Then for each singular perturbation parameter \( \epsilon \in (0, \epsilon'[, \text{ the closed-loop system (11) is asymptotically stable and its } H_\infty \text{ norm is less than or equal to } \gamma'. \\

**Proof.** See Appendix A. □

**Lemma 3** [8]. For a given positive scalar \( \epsilon' \), if the following conditions are satisfied,
\[
\begin{align*}
  a & \geq 0 \\
  ac^2 + be + c & < 0 \\
  c & < 0
\end{align*}
\]
where \( a, b \) and \( c \) are constants, then
\[
ac^2 + be + c < 0, \quad \text{for } \epsilon \in (0, \epsilon']
\]

**Proof.** See Appendix B. □

**Lemma 4** [8]. For a given positive scalar \( \epsilon' \) and matrices \( T_1, T_2, T_3 \), if following conditions are satisfied,
\[
\begin{align*}
  T_1 & \geq 0 \\
  \epsilon^2 T_1 + \epsilon T_2 + T_3 & < 0 \\
  T_3 & < 0
\end{align*}
\]
then
\[
\epsilon^2 T_1 + \epsilon T_2 + T_3 < 0. \quad \text{for } \epsilon \in [0, \epsilon']
\]

**Proof.** See Appendix C. □

### 3. Main results

In this section, a method of evaluating the upper bound of singularly perturbed parameter \( \epsilon \) subject to the stability of the closed-loop system with meeting \( H_\infty \) performance bound requirements is presented. Moreover, two sufficient conditions for designing \( H_\infty \) controllers are given. In particular, one of them can improve the upper bound of the singular perturbation parameter \( \epsilon \) by designing controllers.

#### 3.1. Computation of stability bound of \( \epsilon \) subject to an \( H_\infty \) performance bound constraint

Firstly, the following preliminary lemma is needed.

**Lemma 5.** If there exists a symmetric matrix
\[
Q(\alpha(k)) = \begin{bmatrix} Q_{11}(\alpha(k)) & Q_{12}(\alpha(k)) \\ Q_{21}(\alpha(k)) & Q_{22}(\alpha(k)) \end{bmatrix},
\]
such that the following inequalities hold,
\[
\epsilon^2 \begin{bmatrix}
0 & 0 \\
0 & Q_{22}(\alpha(k))
\end{bmatrix} + \begin{bmatrix}
0 & Q_{22}(\alpha(k)) \\
Q_{21}(\alpha(k)) & 0
\end{bmatrix} + \epsilon \begin{bmatrix}
0 & Q_{22}(\alpha(k)) \\
Q_{21}(\alpha(k)) & 0
\end{bmatrix} < 0, \quad \text{for } \epsilon \in (0, \epsilon']
\]
where
\[
\begin{align*}
\Psi_{11}(k, \alpha(k)) &= \begin{bmatrix} 0 & 0 \\
0 & -S(\alpha(k), \alpha(k + 1)) - S^T(\alpha(k), \alpha(k + 1)) \end{bmatrix} \\
\Psi_{12}(k, \alpha(k)) &= \begin{bmatrix} 0 \ B_0(\alpha(k)K(\alpha(k)))S(\alpha(k), \alpha(k + 1)) \\
0 \ D_{2u}(\alpha(k)K(\alpha(k)))S(\alpha(k), \alpha(k + 1)) \end{bmatrix} \\
\Psi_{41}(k, \alpha(k)) &= \begin{bmatrix} 0 \ 0 \ 0 \ 0 \\
0 \ 0 \ 0 \ 0 \\
0 \ 0 \ 0 \ 0 \\
0 \ 0 \ 0 \ 0 \\
\end{bmatrix}
\end{align*}
\]
then for each singular perturbation parameter \( \epsilon \in (0, \epsilon'] \), the system (9) is asymptotically stable and its \( H_\infty \) norm is less than or equal to \( \gamma' \).
Proof. See Appendix D. □

Based on Lemma 5, a method of evaluating the upper bound of the singular perturbation parameter $\epsilon$ with a prescribed $H_\infty$ performance bound constraint is given in the following theorem.

**Theorem 6.** For a given positive scalar $\epsilon^*$, if there exist matrices $Q^j = (Q^j)^T$, $S^j$, $1 \leq i, j \leq r$

$$Q^j = \begin{bmatrix} Q_{11}^j & (Q_{21}^j)^T \\ Q_{21}^j & Q_{22}^j \end{bmatrix}, \quad S^j = \begin{bmatrix} S_{11}^j & 0 \\ S_{21}^j & S_{22}^j \end{bmatrix}$$

satisfying the following LMIs

\begin{align}
M_{il} < 0, \quad 1 \leq i, 1 \leq l \leq r \\
\frac{1}{r-1} M_{il} + \frac{1}{2} (M_{jl} + M_{jl}) < 0, \quad 1 \leq i \neq j \leq r, \quad 1 \leq l \leq r \\
A_{il} < 0, \quad 1 \leq i, 1 \leq l \leq r \\
\frac{1}{r-1} A_{il} + \frac{1}{2} (A_{jl} + A_{jl}) < 0, \quad 1 \leq i \neq j \leq r, \quad 1 \leq l \leq r
\end{align}

where

\begin{align}
M_{il} &= \begin{bmatrix} Q_{11}^j & 0 \\ 0 & -S^j - (S^j)^T \end{bmatrix} + \begin{bmatrix} 0 & -\gamma I \\ \gamma I & 0 \end{bmatrix} \\
A_{il} &= e^{2 \epsilon} \begin{bmatrix} 0 & 0 & 0 \\ 0 & Q_{22}^j & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e^\epsilon \begin{bmatrix} 0 & (Q_{21}^j)^T \\ Q_{21}^j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{align}

then for each singular perturbation parameter $\epsilon \in (0, \epsilon^*)$, the system (9) is asymptotically stable and its $H_\infty$ norm is less than or equal to $\gamma$.

Proof. See Appendix E. □

**Remark 7.** Theorem 6 presents a method of estimating the upper bound of singularly perturbed parameter $\epsilon$ subject to the stability of the closed-loop system (9) while satisfying an $H_\infty$ performance bound requirement. An upper bound of $\epsilon$ can be obtained by solving the following optimization problem:

Minimize $\epsilon^*$ subject to (24) and (25)

which can be effectively solved by using the LMI Control Toolbox [11].

3.2. $H_\infty$ controller design

In this subsection, two LMI-based methods of designing $H_\infty$ controllers are given. The two methods are with the merit of eliminating the regularity restrictions attached to the Riccati-based solutions. In particular, one of them can improve the upper bound of $\epsilon$ by designing controllers.
3.2.1. Design without the consideration of improving the upper bound of $\epsilon$

In the following, an LMI-based design method without considering the improvement of the upper bound of the singular perturbation parameter $\epsilon$ is given.

**Theorem 8.** If there exist matrices $Q^i = (Q^i)^T$, $S^i$, $L^i$, $1 \leq i, l \leq r$, with

\[
Q^i = \begin{bmatrix}
Q_{11}^i & (Q_{21}^i)^T \\
Q_{21}^i & Q_{22}^i
\end{bmatrix}, \quad S^i = \begin{bmatrix}
S_{11}^i & 0 \\
S_{21}^i & S_{22}^i
\end{bmatrix}, \quad L^i = [L_n^i 0]
\]  

satisfying the following LMIs,

\[
\begin{aligned}
\Gamma_{a_{iil}} &< 0, \quad 1 \leq i, l \leq r \\
\frac{1}{r-1} \Gamma_{a_{iil}} + \frac{1}{2} (\Gamma_{a_{ijl}} + \Gamma_{a_{jil}}) &< 0, \quad 1 \leq i \neq j \leq r, \quad 1 \leq l \leq r
\end{aligned}
\]

where

\[
\Gamma_{a_{ijl}} = \begin{bmatrix}
Q_{11}^i & 0 \\
0 & 0
\end{bmatrix} - S^i - (S^i)^T + * + *
\]

\[
- \gamma l * + *
\]

\[
A^i S^i + B_w^i L^i
\]

\[
B_w^i - Q^i *
\]

\[
C^i S^i + D_{zw}^i L^i
\]

\[
D_{zw}^i 0 - \gamma l
\]

then there exists a sufficient small $\epsilon^* > 0$ such that for $\epsilon \in (0, \epsilon^*)$, the closed-loop system (9) with

\[
K_{i} = L_n S_{11}^{-1}, \quad 1 \leq i \leq r
\]

is asymptotically stable and its $H_\infty$ norm is less than or equal to $\gamma$.

**Proof.** The proof is easily obtained from Theorem 6 and omitted. □

**Remark 9.** Theorem 8 presents a sufficient condition for designing $H_\infty$ controllers for discrete-time fuzzy singularly perturbed systems. The method is based on LMIs, and with the merit of eliminating the regularity restrictions attached to the Riccati-based solutions. Example 12 in Section 4 will illustrate the effectiveness of the method. However, in the method, the issue of improving the upper bound $\epsilon^*$ of the singular perturbation parameter $\epsilon$ is not addressed. As a result, the obtained controller might give a very small stability bound so that the resulting closed-loop system is unstable for a practical singular perturbation parameter $\epsilon$, see Example 14 in Section 4. In order to overcome the difficulty, another new method with the consideration of improving the upper bound of the singular perturbation parameter $\epsilon$ while satisfying $H_\infty$ performance constraints will be proposed in the next subsection.

3.2.2. Design with the consideration of improving the upper bound of $\epsilon$

In this subsection, a new LMI-based $H_\infty$ controller design method with the consideration of improving the upper bound of the singular perturbation parameter $\epsilon$ is given as follows:

**Theorem 10.** For a given positive scalar $\epsilon^*$, if there exist matrices $Q^i = (Q^i)^T$, $S^i$, $L^i$, $1 \leq i, l \leq r$, with

\[
Q^i = \begin{bmatrix}
Q_{11}^i & (Q_{21}^i)^T \\
Q_{21}^i & Q_{22}^i
\end{bmatrix}, \quad S^i = \begin{bmatrix}
S_{11}^i & 0 \\
S_{21}^i & S_{22}^i
\end{bmatrix}, \quad L^i = [L_n^i 0]
\]

satisfying (27b) and the following LMIs,

\[
\begin{aligned}
\Gamma_{b_{iil}} &< 0, \quad 1 \leq i, l \leq r \\
\frac{1}{r-1} \Gamma_{b_{iil}} + \frac{1}{2} (\Gamma_{b_{ijl}} + \Gamma_{b_{jil}}) &< 0, \quad 1 \leq i \neq j \leq r, \quad 1 \leq l \leq r
\end{aligned}
\]

where

\[
\Gamma_{b_{ijl}} = \epsilon^* + \epsilon^* \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
+ \epsilon^* \begin{bmatrix}
0 & (Q_{21}^i)^T \\
0 & 0
\end{bmatrix} + \epsilon^* \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
Q_{11}^i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} - S^i - (S^i)^T + * + *
\]

\[
0 & - \gamma l & * \\
A^i S^i + B_w^i L^i & B_w^i - Q^i & * \\
C^i S^i + D_{zw}^i L^i & D_{zw}^i 0 - \gamma l
\]

Then for $\epsilon \in (0, \epsilon^*)$, the closed-loop system (9) with (28) is asymptotically stable and its $H_\infty$ norm is less than or equal to $\gamma$. 
Proof. The proof is easily obtained from Theorem 6 and omitted.

Remark 11. In this section, Theorem 6 provides an LMI-based condition for estimating the upper bound of singularly perturbed parameter \( \epsilon \) of discrete-time singularly perturbed systems subject to \( H_\infty \) performance bound constraints. In contrast to the existing techniques for estimating stability bounds, the new method can be used to estimate the bound of singularly perturbed parameter \( \epsilon \) subject to \( H_\infty \) performance bound constraints. Moreover, two controller design methods are respectively given by Theorems 8 and 10. The two methods can eliminate the regularity restrictions attached the existing Riccati-based solution [23], the fact is shown by Example 12. In particular, the method given by Theorem 10 can be used to improve the upper bound of singularly perturbed parameter \( \epsilon \) by designing controllers, which is illustrated by Example 14.

4. Example

In Example 12, a discrete-time fuzzy singularly perturbed system is obtained by discretizing a tunnel diode circuit, which is borrowed from [2]. Because the example does not satisfy the regularity property, the conventional Riccati-based method is not applicable. The new LMI-based method given by Theorem 8 will be applied to the example for illustrating its effectiveness.

Moreover, Example 14 is given for better illustrating the effectiveness of the method given by Theorem 10 (i.e., the \( H_\infty \) controller design method with the consideration of improving the upper bound of singularly perturbed parameter \( \epsilon \)).

Example 12. Consider a tunnel diode circuit (Fig. 1), where the diode current is \( i_D(t) \) and the diode voltage \( v_D(t) \), and they satisfy that

\[
\begin{align*}
C \frac{d(v_C(t))}{dt} &= i_C(t), \\
R i_L(t) &= v_R(t) \\
L \frac{d(i_L(t))}{dt} &= v_L(t), \\
i_C(t) &= i_L(t) - i_D(t) \\
u_L(t) &= u(t) - v_R(t) - v_C(t) + w(t)
\end{align*}
\]  

(30)

Let \( x_1(t) = v_C(t) \), \( x_2(t) = i_L(t) \). Combining them and (30), then it follows that

\[
\begin{align*}
C x_1(t) &= 0.2 x_1(t) + 0.05 x_1^2(t) + x_2(t) \\
L x_2(t) &= -x_1(t) - R x_2(t) + u(t) + w(t) \\
z(t) &= x_1(t) + 0.1 w(t)
\end{align*}
\]  

(31)

where \( z(t) \) is the controlled output. Assume that the parameters of the circuit are \( C = 100 \text{ mF}, L = 1 \text{ mH} \) and \( R = 20 \text{ } \Omega \). With these parameters, (31) can be rewritten as follows:

![Fig. 1. The tunnel diode circuit in Example 12.](image-url)
\[ \dot{x}_1(t) = 2x_1(t) + 0.5x_1^2(t) + 10x_2(t) \]
\[ \dot{x}_2(t) = -0.1x_1(t) - 2x_2(t) + 0.1u(t) + 0.1w(t) \]
\[ z(t) = x_1(t) + 0.1w(t) \]

where \( x(t) = [x_1^T(t), x_2^T(t)]^T \) and \( \epsilon = 10^{-4} \). Assume that \( |x(t)| \leq 3 \) and model the nonlinear system (32) by the sector nonlinearity approach in [14], then the following T–S fuzzy model can be obtained:

\[
E_x \dot{x}(t) = \sum_{i=1}^{2} \alpha_i(t) \left( A_i^j x(t) + B_{cu}^j w(t) + B_{du}^j u(t) \right)
\]
\[ z(t) = \sum_{i=1}^{2} \alpha_i(t) \left( C_{cz} x(t) + D_{czw} w(t) \right) \]

where

\[
A_i^j = \begin{bmatrix} 2 & 10 \\ -0.1 & -2 \end{bmatrix}, \quad A_i^j = \begin{bmatrix} 6.9 & 10 \\ -0.1 & -2 \end{bmatrix}, \quad B_{cu}^i = B_{cu}^2 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}
\]
\[
B_{cu}^i = B_{cu}^2 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \quad C_{cz} = C_{cz} = [0, 1] \quad D_{czw} = D_{czw} = 0.1, \quad E_i = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}
\]

and \( \alpha_i(t) \) is the normalized time-varying fuzzy weighting function for each rule \( i = 1, 2 \), satisfies \( \alpha_1(t) = 1 - \frac{x_1^2(t)}{3} \), \( \alpha_2(t) = 1 - \alpha_1(t) \).

From (32), it can be seen that \( x_2(t) \), i.e., the inductor current \( i_L(t) \), changes very fast and exhibits a large range of variation. Therefore, it is very difficult to measure \( x_1(t) \). On the other hand, the slow variable \( x_1(t) \) changes slowly and exhibits a small range of variation, it can be easily measured. Therefore, the slow state feedback controller is needed for this example.

In the sequel, we discretize the model with a sampling period \( T = 0.3 \, \text{s} \) and a zero-order holder, then the following discrete-time singularly perturbed model is obtained:

\[
x(k + 1) = \sum_{i=1}^{2} \alpha_i(k) \left( A_i^j E_j x(k) + B_{du}^j w(k) + B_{du}^j u(k) \right)
\]
\[ z(k) = \sum_{i=1}^{2} \alpha_i(k) \left( C_{dz} x(k) + D_{dzw} w(k) \right) \]

where

\[
A_i^j = \begin{bmatrix} 1.5683 & 7.8415 \\ -0.0784 & -0.3921 \end{bmatrix}, \quad A_i^j = \begin{bmatrix} 6.8210 & 34.1039 \\ -0.3410 & -1.7051 \end{bmatrix}, \quad B_{du}^i = B_{du}^2 = \begin{bmatrix} 0.1894 \\ 0.0405 \end{bmatrix}
\]
\[
B_{du}^i = B_{du}^2 = \begin{bmatrix} 0.1894 \\ 0.0405 \end{bmatrix}, \quad B_{du}^2 = \begin{bmatrix} 0.4547 \\ 0.0273 \end{bmatrix}, \quad C_{dz} = C_{dz} = [1, 0],
\]
\[
D_{dzw} = D_{dzw} = 0.1
\]

Due to \( D_{dzw}(\alpha(k)) \neq 0 \), the regularity property is not satisfied, the conventional Riccati-based methods [23] are not applicable. Applying Theorem 8 (the \( H_\infty \) controller design method without the consideration of improving the upper bound of the singular perturbation parameter \( \epsilon \)), then the following results are obtained:

\[
Q = \begin{bmatrix} 0.4220 & 0.0308 \\ 0.0308 & 995.8465 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.4220 & 0.0293 \\ 0.0293 & 994.0820 \end{bmatrix}
\]
\[
L_1^j = -4.4462, \quad L_2^j = -6.5378, \quad K_1^j = -10.5348, \quad K_2^j = -15.4908
\]
\[
\gamma_{opt} = 0.51639
\]

With the obtained controller gains, the upper bound of the singular perturbation parameter \( \epsilon \) of the tunnel diode circuit system are estimated as \( \epsilon^* = 0.1772 \) by using Theorem 6 under the \( H_\infty \) performance constraint \( \gamma = 0.5614 \). It is bigger than the practical parameter \( \epsilon = 10^{-4} \). Therefore, the designed controller can be used and some simulation results are shown in Figs. 2–4 with an initial state \( x(0) = [0.51]^T \), and \( w(k) = \begin{cases} 1, & 5 \leq k \leq 10 \\ 0, & \text{others} \end{cases} \).

From Figs. 2, 3, and 4, it can be seen that the resulting closed-loop system is asymptotically stable and with a good \( H_\infty \) performance, which further shows the effectiveness of the proposed condition in Theorem 8.

Remark 13. Note that Theorem 8 is applied for designing an \( H_\infty \) controller in Example 12 and the upper bound of singularly perturbed parameter \( \epsilon \) for the resulting closed-loop system is more than the practical \( \epsilon \), hence the controller is applicable. Since the controller design condition in Theorem 8 is without considering the tradeoff of the \( H_\infty \) performance bound \( \gamma \) and the stability bound \( \epsilon \), the obtained controller by Theorem 8 might be not applicable for some singularly perturbed systems. For solving the problem, Theorem 10 can be applied as an alternative, which is shown in Example 14.
Fig. 2. Trajectories of $x(k)$ in Example 12.

Fig. 3. Trajectories of $z(k)$ and $w(k)$ in Example 12.

Fig. 4. Membership functions $z_i(x_1(k))$ for Example 14.
**Example 14.** Consider a numerical example, which is described by (4) with

\[
A^1 = \begin{bmatrix}
0.6 & 1 & -2 & 1 \\
0 & 2 & 0 & -3 \\
0.1 & 0 & 0.3 & 1 \\
2.5 & 0.1 & 0.2 & 0.4
\end{bmatrix}, \quad A^2 = \begin{bmatrix}
0.7 & 1 & -3 & 0.8 \\
0 & 1 & 0 & -2 \\
0.1 & 0 & 0.3 & 1 \\
8 & 0.5 & 0.1 & 0.6
\end{bmatrix}
\]

\[
B^1_w = \begin{bmatrix}
0.2 \\
0 \\
0.2
\end{bmatrix}, \quad B^2_w = \begin{bmatrix}
0.3 \\
0 \\
0.1
\end{bmatrix}, \quad B^1_u = \begin{bmatrix}
0.5 \\
1 \\
0.8
\end{bmatrix}, \quad B^2_u = \begin{bmatrix}
0.3 \\
1 \\
0.4
\end{bmatrix}
\]

\[
C^1_z = C^2_z = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}, \quad D^1_{zw} = \begin{bmatrix}
0 \\
1
\end{bmatrix}, \quad D^2_{zw} = \begin{bmatrix}
0 & 1.2
\end{bmatrix}, \quad D^1_{zu} = D^2_{zu} = \begin{bmatrix}
0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
E_{\epsilon} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \epsilon & 0 \\
0 & 0 & 0 & \epsilon
\end{bmatrix}
\]

where \( \epsilon = 0.05 \) and the membership functions \( x_i(x_k(k)), i = 1, 2 \) are given in Fig. 4.

Applying Theorem 8 to the example (the \( H_\infty \) controller design method without the consideration of improving the upper bound of the singular perturbation parameter \( \epsilon \)), we can obtain the following controller gains and the optimal \( H_\infty \) performance bound.

\[
K^1_a = [-0.1544, -2.1685], \quad K^2_a = [-0.0764, -1.9145]
\]

\[
\gamma_{opt} = 1.3334
\]

\[
\text{Table 1}
\]

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>Theorem 8</th>
<th>Theorem 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>0.0014</td>
<td>0.0527</td>
</tr>
<tr>
<td>2</td>
<td>0.0049</td>
<td>0.0556</td>
</tr>
<tr>
<td>2.5</td>
<td>0.0073</td>
<td>0.0576</td>
</tr>
</tbody>
</table>

\[
\text{Fig. 5. Trajectories of } x(k) \text{ via the gain (33) in Example 14.}
\]
Moreover, by using Theorem 10 with $\epsilon^* = 0.05$ (the $H_\infty$ controller design method with the consideration of improving the upper bound of the singular perturbation parameter $\epsilon$), we can obtain

$$K_1 = \begin{bmatrix} -0.2233 & -2.1834 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -0.7358 & -1.8368 \end{bmatrix}$$

Then the allowable upper bound of the singular perturbation parameter $\epsilon$ of the closed-loop system with the controller gain (33) or (34) can be estimated by using Theorem 6. The obtained results are shown in Table 1.

From Table 1, it can be seen that, for the larger $H_\infty$ performance indices, the larger upper bounds of the singular perturbation parameter $\epsilon$ are achieved by using Theorem 10, which shows that the method of Theorem 10 is effective for improving the upper bound of the singular perturbation parameter $\epsilon$.

Fig. 6. Trajectories of $x(k)$ via the gain (34) in Example 14.

Fig. 7. \[\sqrt{\frac{1}{\sum_{i=1}^{4} x^T(i)z(i)} \sum_{i=1}^{4} w^T(i)w(i)}\] with the gain (34) in Example 14.
Note that $\epsilon = 0.05$, then for the case of $x_1(k) = 0$, $x_2(k) = 1$, the eigenvalues of
\[
A(x(k))E_r + B_u(x(k))K(x(k)) = A^2E_r + B_u^2K^2 = A^2E_r + B_u^2[K^20]
\]
with the controller gain (33) designed by using Theorem 8 are $-1.1190$, $0.6411$, $0.4496$, $-0.1641$, which implies that the resulting closed-loop system is unstable. Therefore, the controller cannot be applied. On the other hand, Table 1 shows the achieved upper bounds of $\epsilon$ of the closed-loop system with the controller gains (34) under different $H_\infty$ performance requirements. The upper bounds are more than $\epsilon = 0.05$. The fact shows that Theorem 10 can be used to effectively improve the upper bound of the singular perturbation parameter $\epsilon$ at the stage of controller design.

What it follows, some simulation results will be given in order to further validate the effectiveness of Theorem 10. Assume that the initial state $x(0) = [-2 \ 8 \ 0 \ 0]^T$ and the disturbance
\[
w(k) = \begin{cases} 
2, & 3 \leq k \leq 10 \\
0, & \text{others}
\end{cases}
\]
Figs. 5 and 6 show the trajectories of the state $x(k)$ of the closed-loop system with the controller gains (33) and (34), respectively. Fig. 7 shows the ratio of $\sum_{i=0}^{k}\varphi_i(i)\xi(i) / \sum_{i=0}^{k}w^T(i)w(i)$ in the simulation.

From Fig. 5, it can be seen that the closed-loop system with the controller gain (33), which is obtained by Theorem 8, is unstable. It can be seen that from Figs. 6 and 7 that the controller gain (34), which is obtained by Theorem 10, can guarantee the stability and meet the $H_\infty$ performance requirement. These simulation results further show the advantage of the method given by Theorem 10, i.e., Theorem 10 can be used to effectively improve the upper bound of the singular perturbation parameter $\epsilon$ at the stage of controller design.

5. Conclusion

In this paper, the $H_\infty$ control problem via slow state variables feedback for discrete-time fuzzy singularly perturbed systems has been investigated. Two LMI-based methods for designing $H_\infty$ controllers via slow state variables feedback are presented, and one of them can be used to improve the upper bound of the singular perturbation parameter $\epsilon$, which can overcome the disadvantage in the conventional design methods where the designed controller might not be used because the resulting allowable upper bound of the singular perturbation parameter is too small. Moreover, a method of evaluating the upper bound of a singular perturbation parameter $\epsilon$ with meeting a prescribed $H_\infty$ performance bound requirement is given in terms of solutions to a set of LMIs. The effectiveness of the proposed methods has been illustrated by the numerical examples.

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Appendix A. Proof of Lemma 2

Proof. Consider the system (11) (which is equivalent to the system (9)), and let $P(x(k)) > 0$. Choose Lyapunov function
\[
V(k) = \gamma^T(k)P(x(k))\xi(k)
\]
then
\[
V(k + 1) - V(k) + z^T(k)\xi(k) - \gamma^2w^T(k)w(k)
\]
\[
= \gamma((A(x(k)) + B_u(x(k))K(x(k)))\xi(k) + B_u(x(k))w(k))^T E_rP(x(k + 1))E_r \times ((A(x(k)) + B_u(x(k))K(x(k)))\xi(k)
\]
\[
+ B_u(x(k))w(k)) - \gamma^2T(k)P(x(k))\xi(k) + ((C_z(x(k)) + D_{zw}(x(k))K(x(z))\xi(k) + D_{zw}(x(z))w(k))^T 
\]
\[
= \gamma\begin{bmatrix} 
\begin{bmatrix} 
\Phi_{11}(k,k +1) & \Phi_{21}(k,k +1)\\ 
\phi_{21}(k,k +1) & \Phi_{22}(k,k +1)
\end{bmatrix} & \begin{bmatrix} 
\xi(k)\\ w(k)
\end{bmatrix}
\end{bmatrix}
\]
where $\phi_{11}(k,k +1), \phi_{21}(k,k +1), \phi_{22}(k,k +1)$ are the same as in (13).

From (12) and the above equality, then it follows that
\[
V(k + 1) - V(k) + z^T(k)\xi(k) - \gamma^2w^T(k)w(k) \leq 0, \quad \text{for } \epsilon \in (0, \epsilon^\star]
\]
From the above inequality, we have that the system (11) with \( \epsilon \in (0, \epsilon^* ] \) is asymptotically stable in the disturbance-free case. For the initial condition \( x(0) = 0 \) and every positive integer \( N \), sum the above inequality from \( k = 0 \) to \( N \), then we can obtain

\[
V(N) - V(0) + \sum_{k=0}^{\infty} z^T(k)z(k) - \gamma^2 \sum_{k=0}^{\infty} w^T(k)w(k) = V(N) + \sum_{k=0}^{\infty} z^T(k)z(k) - \gamma^2 \sum_{k=0}^{\infty} w^T(k)w(k) \leq 0, \text{ for } \epsilon \in (0, \epsilon^* ]
\]

which implies that

\[
\sum_{k=0}^{N} z^T(k)z(k) \leq \gamma^2 \sum_{k=0}^{N} w^T(k)w(k), \text{ for } \epsilon \in (0, \epsilon^* ]
\]

Combining it and Definition 1, it follows that the \( H_\infty \) norm of the closed-loop system (11) is less than or equal to \( \gamma \). \( \square \)

Appendix B. Proof of Lemma 3

**Proof.** Consider the following two cases:

(i): If \( a = 0 \), then from (15) and (16), (17) obviously holds.

(ii): If \( a > 0 \), we consider the following quadratic function of \( \epsilon \),

\[
y(\epsilon) = a\epsilon^2 + b\epsilon + c
\]

(35)

Since \( a > 0 \), \( y(\epsilon) \) is convex function of \( \epsilon \) [4]. From (15) and (16), it follows that \( y(\epsilon^*) < 0 \) and \( y(0) < 0 \), which further implies that \( y(\epsilon) < 0 \) for \( \epsilon \in [0, \epsilon^*] \), i.e., when \( a > 0 \), (17) holds. Thus, the proof is complete. \( \square \)

Appendix C. Proof of Lemma 4

**Proof.** For all nonzero vector \( x(k) \), pre- and post-multiplying (18)–(20) by \( x^T(k) \) and its transpose, then we have

\[
x^T(k)T_1x(k) \geq 0
\]

(36)

\[
\epsilon^2x^T(k)T_1x(k) + \epsilon x^T(k)T_2x(k) + x^T(k)T_3x(k) < 0
\]

(37)

\[
x^T(k)T_3x(k) < 0
\]

(38)

Denote \( a_k = x^T(k)T_1x(k) \), \( b_k = x^T(k)T_2x(k) \), \( c_k = x^T(k)T_3x(k) \). Substituting \( a_k \), \( b_k \) and \( c_k \) into (36)–(38), then it follows that

\[
a_k \geq 0
\]

(39)

\[
a_k \epsilon^2 + b_k \epsilon + c_k < 0
\]

(40)

\[
c_k < 0
\]

(41)

From (39)–(41) and applying Lemma 3, we can obtain \( a_k \epsilon^2 + b_k \epsilon + c_k < 0 \) for \( \epsilon \in [0, \epsilon^*] \), i.e., for all nonzero vector \( x(k) \),

\[
\epsilon^2x^T(k)T_1x(k) + \epsilon x^T(k)T_2x(k) + x^T(k)T_3x(k) < 0, \text{ for } \epsilon \in [0, \epsilon^*]
\]

which implies that matrix inequality (21) holds. Thus, the proof is complete. \( \square \)

Appendix D. Proof of Lemma 5

**Proof.** (22) can be rewritten as follows,

\[
\begin{bmatrix}
\Psi_{11}(k) & * & * & * \\
0 & -\gamma I \\
\Psi_{31}(k) & B_m(x(k)) & -Q(x(k+1)) & * \\
\Psi_{41}(k) & D_m(x(k)) & 0 & -\gamma I
\end{bmatrix} < 0, \text{ for } \epsilon \in (0, \epsilon^* ]
\]

(42)

where \( \Psi_{31}(k), \Psi_{41}(k) \) are the same as in (23) and

\[
\Psi_{11}(k) = E, Q(x(k))E - S(x(k), x(k+1)) - S^T(x(k), x(k+1))
\]

where \( E \) is same as in (5).
From (42), we can obtain that $\Psi_{11}(k) < 0$. Considering the block (3,3) of (42), then yields $Q(\alpha(k)) > 0$. Combining it with $\Psi_{11}(k) < 0$, then we have $S(\alpha(k), \alpha(k + 1)) + S^T(\alpha(k), \alpha(k + 1)) > 0$. Pre- and post-multiply (42) by
\[
\begin{bmatrix}
S^{-T}(\alpha(k), \alpha(k + 1)) & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
S^{-1}(\alpha(k), \alpha(k + 1)) & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{bmatrix}
\]
respectively. Then it follows that
\[
\begin{bmatrix}
\tilde{\Psi}_{11}(k) & * & * \\
0 & -\gamma I & * \\
A(\alpha(k)) + B_u(\alpha(k))K(\alpha(k)) & B_w(\alpha(k)) & -Q(\alpha(k + 1)) + * \\
C_z(\alpha(k)) + D_{zw}(\alpha(k))K(\alpha(k)) & D_{zw}(\alpha(k)) & 0 & -\gamma I
\end{bmatrix}
< 0, \quad \text{for } \epsilon \in (0, \epsilon^*]
\]
where
\[
\tilde{\Psi}_{11}(k) = S^{-T}(\alpha(k), \alpha(k + 1))E_sQ(\alpha(k))E_sS^{-1}(\alpha(k), \alpha(k + 1)) - S^{-1}(\alpha(k), \alpha(k + 1)) - S^{-T}(\alpha(k), \alpha(k + 1))
\]
Let $P^{-1}(\alpha(k)) = E_sQ(\alpha(k))E_s$, therefore,
\[
P(\alpha(k)) > 0
\]
then
\[
\left(S^{-T}(\alpha(k), \alpha(k + 1)) - P(\alpha(k))\right)P^{-1}(\alpha(k))\left(S^{-1}(\alpha(k), \alpha(k + 1)) - P(\alpha(k))\right) \geq 0
\]
which implies that
\[
S^{-T}(\alpha(k), \alpha(k + 1))P^{-1}(\alpha(k))S^{-1}(\alpha(k), \alpha(k + 1)) - S^{-1}(\alpha(k), \alpha(k + 1)) - S^{-T}(\alpha(k), \alpha(k + 1)) \geq -P(\alpha(k))
\]
Combining it with (43), we can obtain
\[
\begin{bmatrix}
-P(\alpha(k)) & * & * & * \\
0 & -\gamma I & * & * \\
A(\alpha(k)) + B_u(\alpha(k))K(\alpha(k)) & B_w(\alpha(k)) & -E_s^{-1}P^{-1}(\alpha(k + 1))E_s^{-1} & * \\
C_z(\alpha(k)) + D_{zw}(\alpha(k))K(\alpha(k)) & D_{zw}(\alpha(k)) & 0 & -\gamma I
\end{bmatrix}
< 0, \quad \text{for } \epsilon \in (0, \epsilon^*]
\]
Applying Schur complement lemma to (45), then we have
\[
\begin{bmatrix}
-P(\alpha(k)) & * & * \\
0 & -\gamma I & * \\
A(\alpha(k)) + B_u(\alpha(k))K(\alpha(k)) & B_w(\alpha(k)) & -E_s^{-1}P^{-1}(\alpha(k + 1))E_s^{-1} \\
C_z(\alpha(k)) + D_{zw}(\alpha(k))K(\alpha(k)) & D_{zw}(\alpha(k)) & 0
\end{bmatrix}
\]
where
\[
\begin{align*}
\mathcal{T}_{11}(k) &= -P(\alpha(k)) + \frac{1}{\gamma} \left( C_z(\alpha(k)) + D_{zw}(\alpha(k))K(\alpha(k)) \right)^T \\
&\quad \times \left( C_z(\alpha(k)) + D_{zw}(\alpha(k))K(\alpha(k)) \right) \\
\mathcal{T}_{21}(k) &= \frac{1}{\gamma} D^T_{zw}(\alpha(k)) \left( C_z(\alpha(k)) + D_{zw}(\alpha(k))K(\alpha(k)) \right) \\
\mathcal{T}_{22}(k) &= -\gamma I + \frac{1}{\gamma} D^T_{zw}(\alpha(k)) D_{zw}(\alpha(k))
\end{align*}
\]
Applying Schur complement lemma to the above inequality, again, then it follows that
\[
\begin{bmatrix}
\mathcal{T}_{11}(k) & \mathcal{T}^T_{21}(k) \\
\mathcal{T}_{21}(k) & \mathcal{T}_{22}(k)
\end{bmatrix}
\begin{bmatrix}
A^T(\alpha(k)) + K^T(\alpha(k))B_u^T(\alpha(k)) \\
B_w^T(\alpha(k))
\end{bmatrix}
E_sP(\alpha(k + 1))E_s \times \left[ A(\alpha(k)) + B_u(\alpha(k))K(\alpha(k)) \right] B_w(\alpha(k)) \]
\[
= \Phi_{11}(k, k + 1) \Phi_{21}^T(k, k + 1) \Phi_{21}(k, k + 1) \Phi_{22}(k, k + 1) \quad < 0, \quad \text{for } \epsilon \in (0, \epsilon^*]
\]
where $\Phi_{11}(k, k + 1)$, $\Phi_{21}(k, k + 1)$ and $\Phi_{22}(k, k + 1)$ are the same as in (13).
Then applying Lemma 2 to (46), we have that, for each singular perturbation parameter \(\epsilon \in (0, \epsilon^*)\), the system (11) is asymptotically stable and its \(H_\infty\) norm is less than \(\gamma\). Then the conclusion follows. \(\square\)

### Appendix E. Proof of Theorem 6

**Proof.** Considering the block (3,3) of (24a) and from (24a), it follows that
\[
Q^i > 0 \quad 1 \leq i \leq r
\]
which also implies that
\[
Q_{22}^i > 0 \quad 1 \leq i \leq r
\]
Let
\[
T_{1i} = \begin{bmatrix}
0 & 0 & 0 \\
0 & Q_{22} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
T_{2i} = \begin{bmatrix}
0 & (Q_{21}^i)^T & 0 & 0 \\
Q_{21} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
T_{3il} = M_{il}
\]
From (47), it follows that
\[
T_{1i} \geq 0
\]
From (25a), we have
\[
\epsilon^2 T_{1i} + \epsilon T_{2i} + T_{3il} < 0
\]
From (24a), we have
\[
T_{3il} < 0
\]
Applying Lemma 4 to (48a), then yields
\[
\epsilon^2 T_{1i} + \epsilon T_{2i} + T_{3il} < 0, \quad \text{for} \ \epsilon \in [0, \epsilon^*], \quad 1 \leq i \leq r
\]
i.e.,
\[
\overline{A}_{il} < 0, \quad \text{for} \ \epsilon \in [0, \epsilon^*], \quad 1 \leq i, l \leq r
\]
where
\[
\overline{A}_{il} = \epsilon^2 \begin{bmatrix}
0 & 0 & 0 \\
0 & Q_{22} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} + \epsilon \begin{bmatrix}
0 & (Q_{21}^i)^T & 0 & 0 \\
Q_{21} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
Q_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} - \begin{bmatrix}
S^i & -S^i & * & * & * \\
0 & 0 & -\gamma I & * & * \\
(A^i + B^i K^i)S^i & B^i - Q^i & * & * & * \\
(C^i + D^i K^i)S^i & D^i - \gamma I & 0 & -\gamma I
\end{bmatrix}
\]
Similarly, from (47), (24b) and (25b), we can also obtain
\[
\frac{1}{r - 1} \overline{A}_{il} + \frac{1}{2} (\overline{A}_{ij} + \overline{A}_{jl}) < 0, \quad \text{for} \ \epsilon \in [0, \epsilon^*], \quad 1 \leq i \neq j \leq r, \quad 1 \leq l \leq r
\]
where
\[
\overline{A}_{ij} = \epsilon^2 \begin{bmatrix}
0 & 0 & 0 \\
0 & Q_{22} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} + \epsilon \begin{bmatrix}
0 & (Q_{21}^i)^T & 0 & 0 \\
Q_{21} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
Q_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} - \begin{bmatrix}
S^i & -S^i & * & * & * \\
0 & 0 & -\gamma I & * & * \\
A^i S^i + B^i K^i S^i & B^i - Q^i & * & * & * \\
C^i S^i + D^i K^i S^i & D^i - \gamma I & 0 & -\gamma I
\end{bmatrix}
\]
Applying the parameterized linear matrix inequality (PLMI) technique in [27] to (49a), then yields
\[
\sum_{i=1}^{r} \sum_{j=1}^{r} x_i(k)x_j(k)\mathcal{A}_{ij} < 0, \quad 1 \leq i \leq r
\]

Multiplying the above inequality by \(x_i(k+1)\) and summing them from \(l = 1\) to \(r\), then we can obtain
\[
\sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{l=1}^{r} x_i(k)x_j(k)x_i(k+1)\mathcal{A}_{ij} < 0
\]  
(50)

Note that
\[
\sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{l=1}^{r} x_i(k)x_j(k)x_i(k+1)\mathcal{A}_{ij} = \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{l=1}^{r} x_j(k)x_i(k+1)\mathcal{A}_{ij} = K(x(k))S_{ii}(x(k), x(k+1))
\]

where \(K(x(k))\) is the same as in (10) and
\[
S(x(k), x(k+1)) = \left[ \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{l=1}^{r} x_i(k)x_j(k+1)\mathcal{S}_{ij} \right]^{1/2} = \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{l=1}^{r} x_i(k)x_j(k+1)\mathcal{S}_{ij}
\]

Then
\[
\sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{l=1}^{r} x_i(k)x_j(k)x_i(k+1)\mathcal{A}_{ij} = B_{ii}(x(k))K(x(k))S(x(k), x(k+1))
\]

Therefore, (50) implies that (22) holds. Combining it and Lemma 5, it follows that for the singular perturbation parameter \(\varepsilon \in (0, \varepsilon^1]\), the system (9) is asymptotically stable and its \(H_\infty\) norm is less than \(\gamma\). Then the conclusion follows. □

References


