Fuzzy Controller Design for Markovian Jump Nonlinear Systems

Jiuxiang Dong and Guang-Hong Yang*

Abstract: This paper is concerned with the problem of state feedback control of continuous-time nonlinear Markovian jump systems, which are represented by Takagi-Sugeno fuzzy models. A new method for designing state feedback stabilizing controllers is presented in terms of solvability of a set of linear matrix inequalities (LMIs), and it is shown that the new design method provides better or at least the same results of the existing method in the literature. Finally, a numerical example is given to illustrate the effectiveness of the proposed method.

Keywords: Fuzzy control, linear matrix inequalities (LMIs), Markovian jump parameters.

1. INTRODUCTION

Many dynamic systems are often inherently vulnerable to component failures or repairs, sudden environmental disturbances, changing subsystem interconnections, abrupt variations of the operating point of a nonlinear plant, etc. Such systems can be modeled as hybrid system with two components in the state vector. The first one which varies continuously is referred to be the continuous state of the systems and the second one which varies discretely is referred to be the model of the system. A special class of hybrid systems is a Markovian jump system, which is composed of many operating models. Switching of the system models is governed by a Markov process with a finite set of states \( N = \{1,2,...,r\} \). The application of Markovian jump systems can be found in manufacturing systems, aircraft control, target tracking, robotics, solar thermal receiver control, and power system [1]. Some important results have been obtained for Markovian jump linear systems, see [2-10] and the references therein. However, the control design of nonlinear Markovian jump systems remains as an open area [25,28].

On the other hand, an important approach to nonlinear control system design is by Takagi and Sugeno (T-S) fuzzy systems, which is composed of local linear time-invariant systems and through membership functions, these local systems are connected smoothly. Therefore, the technique in the conventional linear system theory can be applied to T-S fuzzy systems. In recent years, the T-S fuzzy systems has been intensively studied, and many significant advances have been achieved, see [12-24] and the references therein. In particular, in very recently, fuzzy system with Markovian jump are investigated in [25-28]. Based on linear matrix inequality (LMI) technique, [25] presents a technique for designing a robust \( H_\infty \) output feedback controller for Markovian jump fuzzy systems (MJFS). A sufficient condition for designing \( H_\infty \) filter for MJFS are given in [26]. By considering Markovian jump fuzzy model the approximation error between a fuzzy model and a nonlinear system, sufficient conditions for robust \( H_\infty \) fuzzy control design are given in [27]. In particular, by adding a slack variable to separate system matrices and a stochastic Lyapunov matrix to alleviate the interrelation between the stochastic Lyapunov matrix and system matrices containing controller variables, [28] presents a method for designing state feedback controllers for uncertain MJFSs, which can provide less conservative results than that using a single Lyapunov function for all probable operation modes in [29].

In this paper, we will continue to study the problem of state feedback control of continuous-time nonlinear Markovian jump systems, which are represented by...
Takagi-Sugeno fuzzy models. The main contribution is that a new method for designing state feedback stabilizing controllers is presented in terms of solvability of a set of linear matrix inequalities (LMIs). Compared with the existing method given in [28], the new feature in the technical development is that for reducing the conservatism in stabilizing controller design, more slack variables are introduced in the design conditions, and it is shown that the new design method provides better or at least the same results of the method given in [28]. Furthermore, a numerical example is presented to validate the fact. The paper is organized as follows. In Section 2, system description and some preliminaries are given. Section 3 presents a new LMI-based design method for Markovian jump fuzzy systems, and shows that the new design method provides better or at least the same results of the method given in [28]. The validity of the new result is demonstrated by an example in Section 4. Finally, Section 5 concludes the paper.

2. SYSTEM DESCRIPTION AND PRELIMINARIES

In this paper, we consider a nonlinear Markovian jump system as the following fuzzy system formulation:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(\theta(t)) A_i(\eta(t)) x(t) + \sum_{i=1}^{r} h_i(\theta(t)) B_i(\eta(t)) u(t),$$

(1)

where \( h_i(\theta(t)) \), \( 1 \leq i \leq r \) are fuzzy weighting functions, satisfying

$$h_i(\theta(t)) > 0, \quad i \in S = \{1, 2, \ldots, r\}, \quad \sum_{i=1}^{r} h_i(\theta(t)) = 1.$$

The random form process \( \eta(t) \) is a continuous-time discrete-state Markov process taking values in a finite set \( T = \{1, 2, \ldots, s\} \). The transition rate matrix \( \Pi = [\pi_{kl}]_{k,l \in T} \) and \( \pi_{kl} \) is defined as

$$P[\eta(t + \Delta t) = l | \eta(t) = k] = \begin{cases} \pi_{kl} \Delta t + o(\Delta t) & \text{if } k \neq l, \\ 1 + \pi_{kk} \Delta t + o(\Delta t) & \text{if } k = l, \end{cases}$$

where \( \Delta t > 0, \lim_{\Delta t \to 0} (o(\Delta t) / \Delta t) = 0 \), \( P[\cdot] \) is the probability and \( \pi_{kl} \geq 0 \) for \( k \neq l \), \( \pi_{kk} = -\sum_{l=1, l \neq k}^{r} \pi_{kl} \).

For each possible value of \( \eta(t) = k \), \( k \in T \) in the succeeding discussion, we will denote the matrices associated with the \( k \)-th mode by

$$A_{i,k} = A_i(\eta(t)), \quad B_{i,k} = B_i(\eta(t)).$$

Then the \( k \)-th mode can be considered as follows:

$$\dot{x}(t) = A_k(h)x(t) + B_k(h)u(t),$$

where

$$A_k(h) = \sum_{i=1}^{r} h_i(\theta(t)) A_{i,k}, \quad B_k(h) = \sum_{i=1}^{r} h_i(\theta(t)) B_{i,k}. \quad (2)$$

In this paper, we adopt the following mode-independent fuzzy controller,

$$u(t) = K(h)x(t), \quad (3)$$

where

$$K(h) = \sum_{i=1}^{r} h_i(\theta(t)) K_i. \quad (4)$$

In the following, we recall some existing results for MJFS.

**Lemma 1 [28]:** The MJFS (1) is stochastically stable if there exist matrices \( Q_k = Q_k^T > 0 \), \( k \in T \), such that

$$\begin{bmatrix} A_k(h) + B_k(h)K(h) \end{bmatrix} Q_k + Q_k \begin{bmatrix} A_k(h) + B_k(h)K(h) \end{bmatrix}^T + \sum_{i=1}^{s} \pi_{kl} Q_l < 0, \quad k \in T. \quad (5)$$

**Lemma 2 [28]:** Consider the MJFS (1), for some scalar \( \mu > 0 \), if there exist matrices \( P_k = P_k^T \), \( k \in T \), \( Z \) and \( Y_j, \ j \in S \) satisfying the following LMIs:

$$\Xi_k < 0, \quad i \in S, \ k \in T, \quad \frac{1}{r-1} \Xi_k + \frac{1}{2} (\Xi_{k,ij} + \Xi_{k,jk}) < 0, i \neq j \in S, k \in T, \quad (6)$$

where

$$\Xi_k = \begin{bmatrix} -Z - Z^T & * & * & * \\ A_{i,k} Z + B_{i,k} Y_j + P_k \mu - P_k & * & * & * \\ Z & 0 & -\mu P_k & * \\ 0 & X_k^T & 0 & -P_k \end{bmatrix}, \quad \Xi_{k,ij} = \begin{bmatrix} \sqrt{\pi_{k(i-1)}} P_k \cdots \sqrt{\pi_{k(l-1)}} P_k \sqrt{\pi_{k(l+1)}} P_k \cdots \sqrt{\pi_{k(s-1)}} P_k \end{bmatrix}, \quad \Xi_{k,jk} = \begin{bmatrix} \sqrt{\pi_{k(i-1)}} P_k \cdots \sqrt{\pi_{k(l-1)}} P_k \sqrt{\pi_{k(l+1)}} P_k \cdots \sqrt{\pi_{k(s-1)}} P_k \end{bmatrix}, \quad \Xi_{k,ij} = \begin{bmatrix} \sqrt{\pi_{k(i-1)}} P_k \cdots \sqrt{\pi_{k(l-2)}} P_k \sqrt{\pi_{k(l+1)}} P_k \cdots \sqrt{\pi_{k(s-1)}} P_k \end{bmatrix}, \quad \Xi_{k,ij} = \begin{bmatrix} \sqrt{\pi_{k(i-1)}} P_k \cdots \sqrt{\pi_{k(l-1)}} P_k \sqrt{\pi_{k(l+1)}} P_k \cdots \sqrt{\pi_{k(s-1)}} P_k \end{bmatrix},$$

$$P_k = \text{diag}(P_{1,k}, \ldots, P_{k-1,k}, P_{k+1,k}, \ldots, P_{s,k}), \quad (9)$$

then the MJFS (1) is stochastically stabilizable via the fuzzy controller (3). Furthermore, the state-feedback gain matrices are given by

$$K_j = Y_j Z^{-1}, \quad j \in S.$$
Remark 1: Lemma 2 ([28]) presents an LMI-based method for designing state feedback stabilizing controllers for Markovian jump fuzzy systems. The purpose of this is to derive an less conservative LMI-based stabilizing controller design method, which will be presented in the next section. Moreover, the following existing result is useful for the development of this paper.

Lemma 3 [18]: If the following conditions hold

\[0, 1, 11((i, j) 0, 1, 11, i ≠ j) \leq \mu_i \leq \mu_i, M_{ij} + M_{ji} < 0, \quad 1 \leq i ≠ j \leq r,\]

then the following parameterized linear matrix inequality (PLMI) holds

\[\sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i(\theta_i) \alpha_j(\theta_j) M_{ij} < 0.\]

3. MAIN RESULTS

Theorem 1: Consider the MJFS (1), for some scalar \(\mu > 0\), if there exist matrices \(P_k = P_k^T\), \(S_{ijk} = S_{ijk}^T\), \(Z\), and \(Y_j\), \(i, j \in S, k \in T\), satisfying the following LMIs:

\[\Lambda_{k,ji} < 0, \quad i \in S, k \in T,\]

\[\frac{1}{r-1} \Lambda_{k,ji} + \frac{1}{2}(\Lambda_{k,ji} + \Lambda_{k,ij}) < 0, \quad i ≠ j \in S, k \in T,\]

where

\[\Lambda_{k,ij} = \begin{bmatrix}
-Z - Z^T & * \\
A_{i,k} Z + B_{i,k} Y_j + P_k & \mu^{-1}(S_{ijk} - 2P_k^k) + \pi_{k,k} P_k \\
Z & 0 \\
0 & X_k^T \\
* & * \\
* & * \\
-\mu S_{ijk} & * \\
0 & -P_k
\end{bmatrix},\]

\[X_k = [\sqrt{\pi_{k1} P_k} \ldots \sqrt{\pi_{k(k-1)} P_k} \sqrt{\pi_{k(k+1)} P_k} \ldots \sqrt{\pi_{kk} P_k}],\]

\[P_k = \text{diag}\{P_1, \ldots, P_{k-1}, P_{k+1}, \ldots, P_r\},\]

then the MJFS (1) is stochastically stabilizable via the fuzzy controller (3). Furthermore, the state-feedback gain matrices are given by

\[K_j = Y_j Z^{-1}, \quad j \in S.\]

Proof: From (12) and (13), we have

\[\text{Applying Lemma 3, it follows that (15) holds from (10), (11), and (14).}\]

\[\text{Combining it with (30), it follows that}\]

\[\text{Applying the Schur complement to the above inequality, then it follows that}\]
From (16), it follows that \( Z + Z^T > 0, \) \( P_k > 0, \) \( k \in T. \) Let \( V = Z^{-1}, \) \( Q_k = P_k^{-1}, \) \( k \in T \) nd pre- and post-multiplying (17) by \( \text{diag}[V^T, Q_k] \) and its transpose, then we can obtain

\[
\begin{bmatrix}
-V - V^T + \mu^{-1}S_k^{-1}(h) & * \\
Q_k A_k(h) + Q_k B_k(h) K(h) + V & -\mu^{-1}S_k^{-1}(h) + \sum_{l=1}^{s} \pi_{kl} Q_l
\end{bmatrix} < 0. \tag{17}
\]

Pre- and post-multiplying the above inequality by \( \begin{bmatrix} I & I \end{bmatrix} \) and its transpose, it follows that (5) holds. Then from Lemma 1, we have that the MJFS (1) is stochastically stable. Thus, the proof is complete.

**Remark 2:** In Theorem 1, the PLMI technique in [18] (i.e., Lemma 3) is exploited to derive the design condition such that (30) holds, which is tractable and effective. It should be pointed out that the techniques developed in [17] and [21] are also applicable to render (30) hold, and corresponding design conditions can be obtained, but might be with increasing computational burdens.

Theorem 1 presents a new LMI-based method for designing stabilizing controllers for MJFS. In contrast to the existing method given in [28] (i.e., Lemma 2), the new method given by Theorem 1 can give less or at least the same conservative results than the existing method (Lemma 2), which will be shown in the following theorem.

**Theorem 2:** If the conditions of Lemma 2 hold, then the conditions of Theorem 1 hold.

**Proof:** If the conditions of Lemma 2 hold, then there exist matrices \( P_k = P_k^T, \) \( k \in T, \) and \( Y_j, j \in S \) satisfying (6) and (7). Let \( S_{ijk} = P_k, \) hen from (6) and (7), we have

\[
\begin{align*}
\hat{Z}_{k,ii} &< 0, \quad i \in S, k \in T, \\
\frac{1}{r-1} \hat{Z}_{k,ii} + \frac{1}{2}(\hat{Z}_{k,ii} + \hat{Z}_{k,ji}) &< 0, i \neq j \in S, k \in T, \tag{18}
\end{align*}
\]

where

\[
\hat{Z}_{k,ii} = \begin{bmatrix}
-Z - Z^T & * \\
A_k Z + B_k Y_j + P_k & \mu^{-1}(S_{ijk} - 2P_k) + \pi_{kk} P_k \\
0 & Z \\
0 & X_k^T
\end{bmatrix} < 0
\]

and \( X_k, \) \( P_k \) are same as in (8), (9). From (18) and (19), it follows that (10) and (11) hold, i.e., the conditions of Theorem 1 hold. Thus, the proof is complete.

**Remark 3:** Theorem 2 shows that Theorem 1 can give less or at least the same results than the existing approach in [28] (i.e., Lemma 2). In the next section, a numerical example will be given to validate the fact.

### 4. Example

In the section, a comparis on between the existing method given in [28] (Lemma 2) and the new proposed method given by Theorem 1 is made via a numerical example, and the effectiveness of the new proposed method is illustrated.

The following example is borrowed from [28]. Considering a single-link robot arm:

\[
\dot{\theta}(t) = -\frac{MgL}{J} \sin(\theta(t)) - \frac{D(t)}{J} \dot{\theta}(t) + \frac{1}{J} u(t),
\]

where \( \theta(t) \) is the angle position of the arm, and \( u(t) \) is the control input. \( M \) is the mass of the payload, \( J \) is the moment of inertia, \( g \) is the acceleration of gravity, \( L \) is the length of the arm, and \( D(t) \) is the coefficient of viscous friction. The values of parameters \( g \) and \( L \) are given by \( g = 9.81 \) and \( L = 0.5. \) We assume that the parameter \( D(t) = D \) is time invariant and the parameters \( M \) and \( J \) have three different modes as shown in Table 1. The transition probability-rate matrix that relates the three operation modes is given as follows:

\[
\Pi = \begin{bmatrix}
-0.3 & 0.25 & 0.05 \\
0.1 & -0.2 & 0.1 \\
0.03 & 0.07 & -0.1
\end{bmatrix}
\]

<table>
<thead>
<tr>
<th>Mode ( k )</th>
<th>Parameter ( M )</th>
<th>Parameter ( J )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>

Let \( x_1(t) = \theta(t) \) and \( x_2(t) = \dot{\theta}(t). \) Using the same
procedure as in [13], then the k-th mode can be obtained as follows:

\[
\dot{x}(t) = \sum_{j=1}^{2} h_{j}(x_{j}(t)) A_{i,k} x(t) + \sum_{j=1}^{2} h_{j}(x_{j}(t)) B_{i,k} u(t),
\]

where

\[
A_{11} = \begin{bmatrix} 0 & 1 \\ -gL & -D \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 1 \\ -\beta gL & -D \end{bmatrix},
\]

\[
A_{12} = \begin{bmatrix} 0 & 1 \\ -gL & -0.2D \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0 & 1 \\ -\beta gL & -0.2D \end{bmatrix},
\]

\[
A_{13} = \begin{bmatrix} 0 & 1 \\ -gL & -0.1D \end{bmatrix}, \quad A_{23} = \begin{bmatrix} 0 & 1 \\ -\beta gL & -0.1D \end{bmatrix},
\]

\[
B_{11} = B_{21} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_{12} = B_{22} = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}, \quad B_{13} = B_{23} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},
\]

\[
h_{1}(x_{1}(t)) = \begin{cases} \sin(x_{1}(t)) - \beta x_{1}(t) x_{1}(t)(1 - \beta), & x_{1}(t) \neq 0 \\ 1, & x_{1}(t) = 0 \end{cases},
\]

\[
h_{2}(x_{1}(t)) = \begin{cases} x_{1}(t) - \sin(x_{1}(t)) x_{1}(t)(1 - \beta), & x_{1}(t) \neq 0 \\ 0, & x_{1}(t) = 0 \end{cases},
\]

\[\beta = 10^{-2}/\pi.\]

For the example, both Lemma 2 (the existing method given in [28]) and Theorem 1 are applicable for designing stabilizing controllers. In order to compare the two methods, we assume the one parameter of \(B_{11}\) is uncertain: \(B_{11} = [\delta(t)]\), where \(\delta\) represents the uncertainty. By using Lemma 2 and Theorem 1, we will exploit as big as possible robust bound \(\lambda\) of \(\delta\) such that the condition of Lemma 2 or Theorem 1 holds for \(|\delta| \leq \lambda\). The obtained bounds are given in Table 2.

From Table 2, it can be seen that the new proposed method given by Theorem 1 gives bigger robust bound of \(\delta\), which justifies the result of Theorem 2, where it shows that Theorem 1) can provide less conservative designs than Lemma 2.

### 5. CONCLUSIONS

In this paper, we have studied the problem of state feedback control of continuous-time nonlinear Markovian jump systems, which are represented by Takagi-Sugeno fuzzy models. The main contribution is that a new method for designing state feedback stabilizing controllers is presented in terms of solvability of a set of linear matrix inequalities (LMIs), and it is shown that the new design method provides better or at least the same results of the existing method in the literature. Finally, a numerical example is given to illustrate the effectiveness of the proposed method.

### REFERENCES


Jiuxiang Dong was born in Shenyang, China, on Jan. 2, 1978. He received the B.S. degree in Mathematics and Applied Mathematics and the M.S. degree in Control Science and Engineering from Northeastern University, China, in 2001 and 2004, respectively. He is currently pursing a Ph.D. degree at Northeastern University, China. His research interests include fuzzy control, robust control and fault-tolerant control, and system identification.

Guang-Hong Yang received the B.S. and M.S. degrees in Mathematics from Northeast University of Technology, China, in 1983 and 1986, respectively, and the Ph.D. degree in Control Engineering from Northeastern University, China (formerly, Northeast University of Technology), in 1994. He was a Lecturer/Associate Professor with Northeastern University from 1986 to 1995. He joined the Nanyang Technological University in 1996 as a Postdoctoral Fellow. From 2001 to 2005, he was a Research Scientist/Senior Research Scientist with the National University of Singapore. He is currently a Professor at the College of Information Science and Engineering, Northeastern University. His current research interests include fault-tolerant control, fault detection and isolation, nonfragile control systems design, and robust control. Dr. Yang is an Associate Editor for the International Journal of Control, Automation, and Systems (IJCAS), and an Associate Editor of the Conference Editorial Board of the IEEE Control Systems Society.