Hypersurfaces in lightlike cone

Huili Liu\textsuperscript{a,*}, Seoung Dal Jung\textsuperscript{b}

\textsuperscript{a} Department of Mathematics, Northeastern University, Shenyang 110004, PR China
\textsuperscript{b} Department of Mathematics, Cheju National University, Jeju 690-756, Republic of Korea

Received 2 October 2006; received in revised form 28 April 2007; accepted 13 February 2008
Available online 4 March 2008

Abstract

In this paper, we are concerned with hypersurfaces in \((n + 1)\) dimensional lightlike cone. We will give the fundamental theories and some properties of such hypersurfaces and characterize some of them. Finally, using the moving frame method, we calculate the first and the second variation formulas of the area integral for the hypersurface in lightlike cone.

\textcopyright 2008 Elsevier B.V. All rights reserved.

\textit{MSC:} 53B30; 53A30

\textit{Keywords:} Lightlike cone; Conformal flat; Nondegenerate hypersurface; Hyperquadric; Variation of area integral

1. Introduction

In General Relativity, null submanifolds usually appear to be some smooth parts of the achronal boundaries, for example, event horizons of the Kruskal and Kerr black holes and the compact Cauchy horizons in Taub-NUT spacetime, and their properties are manifested in the proofs of several theorems concerning black holes and singularities. Degenerate submanifolds of Lorentzian manifolds may be useful to study the intrinsic structure of manifolds with degenerate metric and to have a better understanding of the relation between the existence of the null submanifolds and the spacetime metric [7].

Although much has been known about submanifolds (hypersurfaces) of the pseudo Riemannian space forms, there are rather few works on the submanifolds (hypersurfaces) of the pseudo-Riemannian lightlike cone. It should be remarked that a simply connected Riemannian manifold of dimension \(n \geq 3\) is conformally flat if and only if it can be isometrically immersed as a hypersurface of the lightlike cone [3,1,2]. Moreover, from the relations between the conformal transformation group and the Lorentzian group of the \(n\) dimensional Minkowski space \(\mathbb{E}^{n}_1\), and the submanifolds of the \(n\)-dimensional Riemannian sphere \(\mathbb{S}^n\) and the submanifolds of the \((n + 1)\) dimensional lightlike cone \(\mathbb{Q}^{n+1}\), we know that it is important to study submanifolds of the lightlike cone [12,9,8,10,6].

In this paper, we are concerned with hypersurfaces in \((n + 1)\) dimensional lightlike cone. We will give the fundamental theories and some properties of such hypersurfaces and characterize some of them. Finally, we will calculate the first and second variation formulas of the area integral for the hypersurface in lightlike cone with zero mean curvature.

\textsuperscript{*} Corresponding author. Fax: +86 24 83680913.
E-mail addresses: liuhl@mail.neu.edu.cn (H. Liu), sdjung@cheju.ac.kr (S.D. Jung).

0393-0440/$ - see front matter \textcopyright 2008 Elsevier B.V. All rights reserved.
2. Preliminaries

Let $\mathbb{E}_q^m$ be the $m$ dimensional pseudo Euclidean space with the metric

$$\bar{g}(x, y) = \langle x, y \rangle = \sum_{i=1}^{m-q} x_i y_i - \sum_{j=m-q+1}^m x_j y_j,$$

where $x = (x_1, x_2, \ldots, x_m)$, $y = (y_1, y_2, \ldots, y_m) \in \mathbb{E}_q^m$. The space $\mathbb{E}_q^m$ is a flat pseudo Riemannian manifold of signature $(m - q, q)$.

Let $M^r$ (or simply $M$) be a submanifold of $\mathbb{E}_q^m$. If the pseudo Riemannian metric $\bar{g}$ of $\mathbb{E}_q^m$ induces a pseudo Riemannian metric $g$ (respectively, a Riemannian metric, a degenerate quadric form) on $M^r$, then $M^r$ is called a timelike (respectively, spacelike, degenerate) submanifold of $\mathbb{E}_q^m$ [6].

Let $c$ be a fixed point in $\mathbb{E}_q^m$ and $r > 0$ be a constant. The pseudo Riemannian sphere is defined by

$$S_q^n(c, r) = \{ x \in \mathbb{E}_q^{n+1} : \bar{g}(x - c, x - c) = r^2 \};$$

the pseudo Riemannian hyperbolic space is defined by

$$H_q^n(c, r) = \{ x \in \mathbb{E}_q^{n+1} : \bar{g}(x - c, x - c) = -r^2 \};$$

the pseudo Riemannian lightlike cone (quadric cone) is defined by

$$Q_q^n(c) = \{ x \in \mathbb{E}_q^{n+1} : \bar{g}(x - c, x - c) = 0 \}.$$

It is well known that the space $S_q^n(c, r)$ is a complete pseudo Riemannian hypersurface of signature $(n - q, q)$, $q \geq 1$, in $\mathbb{E}_q^{n+1}$ with constant sectional curvature $r^{-2}$; the space $H_q^n(c, r)$ is a complete pseudo Riemannian hypersurface of signature $(n - q, q)$, $q \geq 1$, in $\mathbb{E}_q^{n+1}$ with constant sectional curvature $-r^{-2}$; the space $Q_q^n(c)$ is a degenerate hypersurface in $\mathbb{E}_q^{n+1}$. The spaces $\mathbb{E}_q^n, S_q^n(c, r)$ and $H_q^n(c, r)$ are called pseudo Riemannian space forms. The point $c$ is called the center of $S_q^n(c, r), H_q^n(c, r)$ and $Q_q^n(c)$. When $c = 0$ and $q = 1$, we simply denote $Q_q^1(0)$ by $\mathbb{P}^n$ and call it the lightlike cone (or simply the light cone) [4,5,11].

Let $x : M^r \rightarrow \mathbb{E}_q^{n+1} \subset \mathbb{E}_q^{n+2}$ ($n \geq 2$) be a nondegenerate spacelike hypersurface and $\{e_i\}$ be a local orthonormal basis of $T(M^r)$ such that $\langle e_i, e_j \rangle = \delta_{ij}$ (In this section and next section, we use $1 \leq i, j, k, l, \ldots \leq n$ and $1 \leq \alpha, \beta, \gamma, \tau, \ldots \leq n + 2$). Since $\langle x, x \rangle = 0$, we can choose $y$ such that $\langle x, y \rangle = \langle y, y \rangle = 0$, $\langle x, y \rangle = 1$ and

$$\{x, y\} \in T(M^r).$$

From $\langle x, x \rangle = \langle y, y \rangle = 0$ we get $\langle x, dx \rangle = \langle y, dy \rangle = 0$. Therefore, we have

$$\langle e_i, x \rangle = \langle e_i, y \rangle = 0.$$

The conditions $\langle x, y \rangle = 1$ and $\langle e_i, x \rangle = 0$ yield that

$$\langle x, e_i(y) \rangle = -\langle e_i, x \rangle, y = 0.$$

Let $g = (dx, dx) = g_{ij} dx^i dx^j$. We denote the connections of $M$ and $\mathbb{E}_q^{n+2}$ by $\nabla$ and $\bar{\nabla}$, $\bar{\nabla}_i = \bar{\nabla}_{e_i}$. Then the structure equations of $x$ can be written as follows:

$$\langle e_i(x), e_j(x) \rangle = g_{ij} = g_{ji},$$

$$\bar{\nabla}_i y = e_i(y) = A_{ij} g^{ja} e_a(x),$$

$$\bar{\nabla}_j (\bar{\nabla}_i x) = e_j(e_i(x)) = \Gamma^k_{ij} e_k(x) - A_{ji} x - g_{ij} y.$$

From $\langle e_j(e_i(y)), e_j(y) \rangle = e_i(e_j(y))$ we have

$$-A_{j\beta} g^{\alpha \beta} A_{\alpha a} = -A_{j\beta} g^{\alpha \beta} A_{\alpha a},$$

$$-A_{ij} = -A_{ji},$$

$$\partial_j (A_{j\beta} g^{\alpha \beta} y) + A_{j\beta} g^{\alpha \beta} \Gamma^\gamma_{j\alpha} = \partial_j (A_{j\beta} g^{\alpha \beta} y) + A_{j\beta} g^{\alpha \beta} \Gamma^\gamma_{j\alpha}.$$
By (2.6) we get
\[ A_{ik, j} := \nabla_j A_{ik} = \nabla_i A_{jk} := A_{jk, i}. \] (2.7)
The Eq. (2.3) can be written as:
\[ \nabla_i e_k(x) = \delta_i (e_k(x)) - T^a_{ik} e_a = -A_{ik} x - g_{ik} y. \]
From \( e_j(e_i(e_k(x))) = e_i(e_j(e_k(x))) \) we have
\[ \nabla_j \nabla_i (e_k(x)) - \nabla_i \nabla_j (e_k(x)) = -A_{ik} e_j(x) - g_{ik} A_{jk} g^{a\beta} e_\beta(x) + A_{jk} e_i(x) + g_{jk} A_{ia} g^{a\beta} e_\beta(x) \]
\[ = e_\beta(x) (A_{jk} \delta_i^\beta - A_{ik} \delta_j^\beta + g_{jk} A_{ia} g^{a\beta} - g_{ik} A_{ja} g^{a\beta}). \]
By the definition of the Riemannian curvature tensor:
\[ R_{ijkl} := (\nabla_j \nabla_i (e_k(x)) - \nabla_i \nabla_j (e_k(x)), e_l(x)) \]
we get
\[ R_{ijkl} = A_{ik} g_{jl} + A_{jl} g_{ik} - A_{jk} g_{il} - A_{il} g_{jk}. \] (2.8)
Therefore, the integrability conditions of \( x \) are:
\[ A_{ij} = A_{ji}, \] (2.9)
\[ A_{ij, k} = A_{ik, j}, \] (2.10)
\[ R_{ijkl} = A_{ik} g_{jl} + A_{jl} g_{ik} - A_{jk} g_{il} - A_{il} g_{jk}. \] (2.11)
From (2.11) we get the Ricci curvature tensor \( R_{ij} \) and the normalized scalar curvature \( \kappa \) as the follows:
\[ R_{ik} = (n - 2) A_{ik} + (\text{trace } A) g_{ik}, \] (2.12)
\[ n(n - 1) \kappa = 2(n - 1) (\text{trace } A). \] (2.13)

3. Hypersurfaces in lightlike cone

In this section we will give some properties of hypersurfaces in lightlike cone \( \mathbb{Q}^{n+1} \).

Proposition 3.1. Let \( x : \mathbb{M}^n \to \mathbb{Q}^{n+1} \subset \mathbb{P}^{n+2} \) be any nondegenerate hypersurface in lightlike cone \( \mathbb{Q}^{n+1}, n > 2 \), then \( x \) is conformal flat.

Proof. From the definition of the conformal curvature tensor \( W_{ijkl} \):
\[ W_{ijkl} := R_{ijkl} - \frac{1}{n - 2} (R_{ik} g_{jl} - R_{jk} g_{il} + R_{jl} g_{ik} - R_{il} g_{jk}) + \frac{n k}{n - 2} (g_{ik} g_{jl} - g_{jk} g_{il}) \] (3.1)
and (2.11)–(2.13), by a direct calculation, we get
\[ W_{ijkl} \equiv 0. \]
Therefore, any nondegenerate hypersurface \( x \) in \( \mathbb{Q}^{n+1} (n > 2) \) is conformal flat. \( \square \)

By (2.3) we have
\[ \Delta x = g^{ij} \nabla_j \nabla_i x = g^{ij} \nabla_j e_l(x) = g^{ij}(-A_{ij} x - g_{ij} y) = -(\text{trace } A) x - n y. \] (3.2)
From (2.2) and (2.3) we get
\[ \Delta y = e_j (\text{trace } A) g^{ij} e_l(x) - \|A\|^2 x - (\text{trace } A) y, \] (3.3)
where
\[ \|A\|^2 := A_{ij} A^{ij} = A_{ij} A_{kl} g^{ik} g^{jl}. \]
From (3.2) and (3.3) we obtain
\[ \langle \Delta x, \Delta x \rangle = 2n(\text{trace } A), \tag{3.4} \]
\[ \langle \Delta y, \Delta y \rangle = \| \nabla \text{trace } A \|^2 + 2\| A \|^2(\text{trace } A). \tag{3.5} \]
From (3.2) and (3.4) we have
\[ y = -\frac{1}{n} \Delta x - \frac{1}{2n^2} \langle \Delta x, \Delta x \rangle x. \tag{3.6} \]

**Definition 3.2.** Let \( x : M^n \to \mathbb{Q}^{n+1} \hookrightarrow \mathbb{E}^{n+2}_1 \) be a nondegenerate hypersurface. If \( \text{rank } A = n \), then \( y : M^n \to \mathbb{Q}^{n+1} \hookrightarrow \mathbb{E}^{n+2}_1 \) also defines a nondegenerate hypersurface. \( y \) is called the associated hypersurface of \( x \).

**Definition 3.3.** Let \( x : M^n \to \mathbb{Q}^{n+1} \hookrightarrow \mathbb{E}^{n+2}_1 \) be a nondegenerate hypersurface. Define
\[ nH := \langle \Delta x, y \rangle = -\text{trace } A. \tag{3.7} \]
Then \( H \) is called the mean curvature of the nondegenerate hypersurface \( x \). If \( H \equiv 0 \), \( x \) is called the maximal hypersurface.

Let \( y : M^n \to \mathbb{Q}^{n+1} \) be the associated hypersurface of \( x \). We denote the quantities of \( y \) by “tilde”. From (2.2) we can get the metric \( \tilde{g} \) of \( y \) as the follows:
\[ \tilde{g} := \langle dy, dy \rangle = \langle e_i(y), e_j(y) \rangle dx^i dx^j = A_{ij} g^{ij} dx^i dx^j, \]
\[ \tilde{g}_{ij} = A_{ij} A_{j\beta} g^{\alpha\beta}. \tag{3.8} \]
When \( \text{rank}(A) = n \), we have \( T(x(M)) = T(y(M)) \). Then we may assume that
\[ \tilde{\Delta} y = ax + by. \]

From (2.1)–(2.3) and the relations between \( x \) and \( y \) we obtain
\[ a = \langle \tilde{\Delta} y, y \rangle = \tilde{g}^{ij} \langle \nabla_j \tilde{y}_i, y \rangle = \tilde{g}^{ij} \langle \tilde{\nabla}_j e_i(y), y \rangle = -\tilde{g}^{ij} \langle e_i(y), e_j(y) \rangle = -\tilde{g}^{ij} A_{ij} = -n, \]
\[ b = \langle \tilde{\Delta} y, x \rangle = \tilde{g}^{ij} \langle \nabla_j e_i(y), x \rangle = -\tilde{g}^{ij} \langle e_i(y), e_j(x) \rangle = -\tilde{g}^{ij} A_{ij} g^{\alpha\beta} g_{\beta j} = -\tilde{g}^{ij} A_{ij} = -\text{trace } (A^{-1}). \]
Therefore
\[ \tilde{\Delta} y = -nx - (\text{trace } (A^{-1})) y. \tag{3.9} \]
Since
\[ \langle \tilde{\Delta} y, \tilde{\Delta} y \rangle = 2n(\text{trace } (A^{-1})), \]
we get
\[ -\frac{1}{n} \tilde{\Delta} y - \frac{1}{2n^2} \langle \tilde{\Delta} y, \tilde{\Delta} y \rangle y = -\frac{1}{n} (-nx - (\text{trace } (A^{-1})) y) - \frac{1}{2n^2} (2n(\text{trace } (A^{-1}))) y \]
\[ = x + \frac{1}{n} (\text{trace } (A^{-1})) y - \frac{1}{n} (\text{trace } (A^{-1})) y \]
\[ = x. \tag{3.10} \]
Therefore, we have proved the following theorem:

**Theorem 3.4.** Let \( x : M^n \to \mathbb{Q}^{n+1} \hookrightarrow \mathbb{E}^{n+2}_1 \) be a nondegenerate hypersurface, \( y : M^n \to \mathbb{Q}^{n+1} \) be the associated hypersurface of \( x \). Then the associated hypersurface of \( y \) is \( x \).
Theorem 3.5. Let \( x : M^n \to \mathbb{Q}^{n+1}_{E_1} \) be a nondegenerate hypersurface. If \( A_{ij} = \lambda g_{ij} \) for some \( \lambda \in C^\infty(M) \), then \( x \) lies in a hyperquadric.

Proof. Let \( A_{ij} = \lambda g_{ij} \). Then the condition \( A_{ij,k} = A_{ik,j} \) yields that \( \lambda = \text{constant} \). From (2.2) we have
\[
\lambda = (n - 1)\kappa - \text{trace} A.
\]
Therefore
\[
y = \lambda x + c,
\]
where \( c \) is constant vector. By \( (x, y) = 1 \) we get
\[
(c, x) = (y, x) = 1.
\]
(4.1) \( (x, y) = 0 \) and \( (x, c) = 1 \) mean that \( x \) lies in a hyperquadric. From (3.11) we have also
\[
(y, c) = -2\lambda.
\]

Corollary 3.6. Let \( x : M^n \to \mathbb{Q}^{n+1}_{E_1} \) be a nondegenerate hypersurface. If \( A_{ij} = \lambda g_{ij} \) for some \( 0 \neq \lambda \in C^\infty(M) \), then the associated hypersurface \( y \) of \( x \) lies also in a hyperquadric.

Corollary 3.7. Let \( x : M^n \to \mathbb{Q}^{n+1}_{E_1} \) be a nondegenerate hypersurface \( n > 2 \). If \( x \) is an Einstein hypersurface, then \( x \) lies in a hyperquadric.

Proof. If \( x \) is an Einstein hypersurface, by (2.12) we know that \( A_{ij} = \lambda g_{ij} \) for
\[
\lambda = \frac{(n - 1)\kappa - \text{trace} A}{n - 2}.
\]
From Theorem 3.5 we get that \( x \) lies in a hyperquadric.

Theorem 3.8. Let \( x : M^n \to \mathbb{Q}^{n+1}_{E_1} \) be a nondegenerate hypersurface, rank \( (A) = n \). If \( n = 2 \), then \( x \) is maximal if and only if the associated hypersurface \( y \) of \( x \) is also maximal.

Proof. From Definition 3.3 and formulas (3.2) and (3.9), we know that \( x \) is maximal if and only if trace \( A = 0 \); \( y \) is maximal if and only if trace \( A^{-1} = 0 \). If \( n = 2 \), by a direct computation, we obtain that trace \( A = 0 \) if and only if trace \( A^{-1} = 0 \).

4. The variation of area integral

In this section, by using the moving frame method, we calculate the first and the second variation formulas for the area integral of the hypersurface in \( \mathbb{Q}^{n+1}_{E_1} \). Let \( x : M^n \to \mathbb{Q}^{n+1}_{E_1} \) be a nondegenerate hypersurface with boundary \( \partial M \). We consider the variation \( f : M \times (-\varepsilon, \varepsilon) \to \mathbb{Q}^{n+1}_{E_1} \) (\( \varepsilon \) is a positive constant) such that: (i) for each \( t \), \( x_t := f(t) \) \( M^n \to \mathbb{Q}^{n+1}_{E_1} \) is a hypersurface; (ii) \( x_0 = x \) on \( M^n \); (iii) \( x_t = x \) and \( d\lambda_t(TM^n) = d\lambda(TM^n) \) on \( \partial M \) for each \( t \). In the following, we use that
\[
i, j, k, l, \ldots = 1, 2, \ldots, n
\]
\[
\alpha, \beta, \gamma, \tau, \ldots = n + 1, n + 2
\]
\[
A, B, C, D, \ldots = 1, 2, \ldots, n, n + 1, n + 2.
\]
Let \( \{e_i\} \) and \( \{e_\alpha\} \) be the local orthonormal basis of \( T(M^n, x^+_i g) \) and \( T^+(M^n, x^+_i g) \) with the dual basis \( \{\omega^i\} \) and \( \{\omega^\alpha\} \) as in Section 2 for the hypersurface in \( \mathbb{Q}^{n+1}_{E_1} \). Then \( \{dx_t(e_i), e_\alpha\} \) is a local orthonormal basis for \( E_{1}^{n+2} \) along \( x_t \). We denote the extended local orthonormal basis and dual basis also by \( \{e_i, e_\alpha\} \) and \( \{\omega^i, \omega^\alpha\} \) (or by \( \{e_i, \tilde{e}_\alpha\} \) and \( \{\omega^i, \tilde{\omega}^\alpha\} \) to avoid the confusion). Then
\[
dx = \omega^A e_A = \omega^i e_j + \omega^\alpha e_\alpha = \omega^i e_j,
\]
\[
d\omega_t = \omega^A e_A = \omega^i e_j + \omega^\alpha e_\alpha,
\]
\[
de_\alpha = \omega^A e_A = \omega^\alpha e_j + \omega^\beta e_\beta.
\]
From $dx = de_{n+1}$, we have
\[ \omega^i e_j = \omega^j_{n+1} e_j + \omega^\alpha_{n+1} e_\alpha. \]

Hence
\[ \begin{align*}
\omega^j_{n+1} & = \omega^j, \\
\omega^\alpha_{n+1} & = 0.
\end{align*} \tag{4.4} \tag{4.5} \]

The relations $\langle e_j, e_{n+1} \rangle = 0$ and $\langle e_i, e_{n+2} \rangle = 0$ yield that
\[ \begin{align*}
\omega^{n+2}_i & + \omega^j_{n+1} g_{ij} = 0, \tag{4.6} \\
\omega^{n+1}_j & + \omega^j_{n+2} g_{ij} = 0. \tag{4.7}
\end{align*} \]

From $\langle e_\alpha, e_\beta \rangle = 1 - \delta_{\alpha\beta}$ we have $\omega^{n+2}_n = \omega^{n+1}_n = \omega^{n+1}_n + \omega^{n+2}_n = 0$. Then with (4.5) we get
\[ \omega^{n+1}_n = \omega^{n+2}_n = \omega^{n+2}_n = 0. \tag{4.8} \]

From $d\omega^A = \omega^B \wedge \omega^A_B$ and $\omega^\alpha = 0$, when $\omega^\alpha$ is restricted to $M^n$, we have
\[ \begin{align*}
d\omega^j = & \omega^A \wedge \omega^j_A = \omega^j \wedge \omega^j_j + \omega^\alpha \wedge \omega^j_j = \omega^j \wedge \omega^j_j, \\
d\omega^\alpha = & \omega^A \wedge \omega^\alpha_A = \omega^j \wedge \omega^j_j + \omega^\beta \wedge \omega^\alpha_j = \omega^j \wedge \omega^j_j = 0. \tag{4.9} \tag{4.10}
\end{align*} \]

We assume that
\[ \omega^i_j = A^\alpha_{ij} \omega^j. \tag{4.11} \]

From (4.4) and (4.6) we have $\omega^{n+2}_n = -\omega^j_{n+1} g_{ij} = -\omega^j g_{ij} = A^{n+2}_{ij} \omega^j$. Then
\[ A^{n+2}_{ij} = -g_{ij}. \tag{4.12} \]

We put
\[ A^{n+1}_{ij} = A_{ij}. \tag{4.13} \]

From
\[ d\omega_{AB} = \omega^C_A \wedge \omega_{CB} - \frac{1}{2} R_{ABCD} \omega^C \wedge \omega^D = \omega^C_A \wedge \omega_{CB} \tag{4.14} \]

we have
\[ d\omega_{ij} = \omega^A_i \wedge \omega_{Aj} = \omega^j_i \wedge \omega_{kj} + \omega^\alpha_j \wedge \omega_{\alpha j}. \tag{4.15} \]

Then
\[ -\frac{1}{2} R_{ijkl} \omega^k \wedge \omega^l = d\omega_{ij} - \omega^j_i \wedge \omega_{kj} = \omega^\alpha_i \wedge \omega_{\alpha j} \\
= \omega^\alpha_i \wedge \omega^A_{j} g_{\alpha a} = \omega^\alpha_i \wedge \omega^\beta_{j} g_{\beta a} \\
= \omega^{n+1}_i \wedge \omega^j_{n+2} + \omega^j_{n+2} \wedge \omega^{n+1}_i \\
= A^{n+1}_{ik} \omega^k \wedge A^{n+2}_{jl} \omega^j + A^{n+2}_{ik} \omega^k \wedge A^{n+1}_{jl} \omega^j \\
= -A_{ik} g_{jl} \omega^k \wedge \omega^j - A_{jl} g_{ik} \omega^k \wedge \omega^j \\
= -(A_{ik} g_{jl} + A_{jl} g_{ik}) \omega^k \wedge \omega^j. \]

Therefore, we get the integrability condition (2.11)
\[ R_{ijkl} = A_{ik} g_{jl} + A_{jl} g_{ik} - A_{jl} g_{jk} - A_{jk} g_{il}. \]
From
\[ \text{d} \omega_i^A = \omega_i^A \wedge \omega_{Aa} = \omega_i^j \wedge \omega_{jA} + \omega_i^\beta \wedge \omega_{\beta A} = \omega_i^j \wedge \omega_{jA} = \omega_i^j \wedge \omega_j^\beta g_{\beta A} \]  
we have
\[ \text{d} \omega_i^{(n+2)} = \omega_i^j \wedge \omega_j^{n+1} = \omega_i^j \wedge A_{jk} \omega_k. \]

Then
\[ 0 = \text{d} \omega_i^{(n+2)} - A_{jk} \omega_i^j \wedge \omega_k = \text{d} \omega_i^{n+1} - A_{jk} \omega_i^j \wedge \omega_k \]
\[ = \text{d}(A_{ik} \omega^k) - A_{jk} \omega_i^j \wedge \omega_k = \text{d}A_{ik} \wedge \omega^k + A_{ik} \text{d}\omega^k - A_{jk} \omega_i^j \wedge \omega_k \]
\[ = \text{d}A_{ik} \wedge \omega^k - A_{ij} \omega_i^j \wedge \omega_k - A_{jk} \omega_i^j \wedge \omega_k \]
\[ = (\text{d}A_{ik} - A_{ij} \omega_i^j - A_{jk} \omega_i^j) \wedge \omega^k \]
\[ = A_{ik} \omega_i^j \wedge \omega^k. \]

Therefore, we get the integrability condition (2.10)
\[ A_{ik,l} = A_{il,k}. \]

From \( \text{d} \omega_{A} = \omega_{A}^A \wedge \omega_{Aa} \) we can easily get the integrability condition (2.9)
\[ A_{ij} = A_{ji}. \]

Since any 1-form \( \Omega \) on \( M^n \times \{-\varepsilon, \varepsilon \} \) can be written as \( \Omega = \omega + v \text{d}t \), where \( \omega \) is the 1-form on \( M^n \) and \( v \) the smooth function on \( M^n \times \{-\varepsilon, \varepsilon \} \), then we assume that
\[ f^* A^A = \omega^A + v^A \text{d}t, \]
\[ f^* \omega_{AB} = \omega_{AB} + \phi_{AB} \text{d}t. \]

By \( v^A = f^* \omega^A (\frac{\partial}{\partial t}) = \tilde{\omega}^A (\frac{\partial f}{\partial t}) \), we know that when \( t = 0 \)
\[ V := \frac{\partial f}{\partial t} = \sum_i v^i \text{d}x_i(e_i) + \sum_a v^a e_a := V^T + V^N \]
is the variation vector field of \( x_t \). Since \( \{e_i, \frac{\partial}{\partial t} \} \) is a local basis of \( T(M^n \times \{-\varepsilon, \varepsilon \}) \), from (4.17) and \( df(e_i) = dx_i(e_i) \), we get
\[ \omega^A(e_i) = f^* \omega^A(e_i) = \tilde{\omega}^A(\text{d}f(e_i)) = 0. \]

Then
\[ x_t^* \omega^A = \omega^A \equiv 0, \quad x_t^* \omega^j = \omega^j, \]
\[ g_t = x_t^* g = \sum_i \omega^i \otimes \omega^i. \]

Since the exterior differential operator \( \text{d} \) on \( T^*(M \times \mathbb{R}) = T^*M \oplus T^*\mathbb{R} \) is given by
\[ \text{d} = \text{d}_M + \text{d}t \wedge \frac{\partial}{\partial t}, \]
then from
\[ \text{d}(\omega^j + v^j \text{d}t) = (\omega^j + v^j \text{d}t) \wedge (\omega^j + \phi^j \text{d}t) + (\omega^A + v^A \text{d}t) \wedge (\omega^i + \phi^i \text{d}t) \]
\[ = \omega^j \wedge \omega^j + \phi^j \wedge \omega^j \wedge \text{d}t + v^j \text{d}t \wedge \omega^j + v^A \text{d}t \wedge \omega^A, \]
\[ d(\omega^j + v^j \, dt) = \left( dM + dr \wedge \frac{\partial}{\partial t} \right)(\omega^j + v^j \, dt) \]
\[ = dM \omega^j + dM v^j \wedge dr + dr \wedge \frac{\partial \omega^j}{\partial t}, \]
we get
\[ dM \omega^j = \omega^j \wedge \omega^j , \quad (4.20) \]
\[ \frac{\partial \omega^j}{\partial t} = dM v^j - \phi^j_i \omega^j + v^j \omega^j_i + v^\alpha \omega^j_i. \quad (4.21) \]
From
\[ d(v^\alpha \, dt) = (\omega^j + v^j \, dt) \wedge (\omega^\alpha_i + \phi^\alpha_i \, dt) + \omega^\beta_j + \phi^\beta_j \, dt) (\omega^\alpha_i + \phi^\alpha_i \, dt) \]
\[ = \omega^j \wedge \omega^\alpha_i + \phi^\alpha_i \omega^j \wedge dr + v^j_\alpha \wedge \omega^\alpha_i + v^\beta \wedge \omega^\alpha_i, \]
\[ d(v^\alpha \, dt) = \left( dM + dr \wedge \frac{\partial}{\partial t} \right)(v^\alpha \, dt) = dM v^\alpha \wedge dr, \]
we get
\[ \omega^j \wedge \omega^\alpha_i = 0, \]
\[ dM v^\alpha = \phi^\alpha_i \omega^j - v^j \omega^\alpha_i - v^\beta \omega^\alpha_i \]
\[ = \phi^\alpha_i \omega^j - v^j A^\alpha_i \omega^j, \]
\[ dM v^\alpha = v^\alpha_i \omega^j, \]
that is
\[ 0 = \omega^j \wedge \omega^\alpha_i, \quad (4.22) \]
\[ v^\alpha_i = \phi^\alpha_i - v^j A^\alpha_i. \quad (4.23) \]
Therefore using (4.12), (4.13) and (4.21) we have
\[ \frac{\partial}{\partial t}(\omega^1 \wedge \cdots \wedge \omega^n) = (v^j_i + v^\alpha_i A^\beta_i g_{\alpha \beta}) \omega^1 \wedge \cdots \wedge \wedge \omega^n \]
\[ = (v^j_i + v^{n+1} A^\beta_i g^{(n+1) \beta} + v^{n+2} A^{(n+2) \alpha} \omega^1 \wedge \cdots \wedge \omega^n \]
\[ = (v^j_i + v^{n+1} A^{(n+2) \alpha} + v^{n+2} A^{(n+1) \alpha} \omega^1 \wedge \cdots \wedge \wedge \omega^n \]
\[ = (v^j_i + A^\alpha_i (V_N, x)) \omega^1 \wedge \cdots \wedge \wedge \wedge \omega^n - n \langle V_N, y \rangle \omega^1 \wedge \cdots \wedge \wedge \omega^n. \]
(4.24)
We use \[ dM = \omega^1 \wedge \cdots \wedge \wedge \omega^n, \quad A^\alpha_i = A^\alpha_i g^{ki} \] and assume that \[ \frac{\partial}{\partial t} \equiv 0 \] on \[ dM. \] Then we have the first variation formula of the area integral for the hypersurface \[ x : M^i \rightarrow Q^n + 1 \rightarrow E^{n+2}, \]
\[ V'(0) = \frac{d}{dt} \left( Vol(M) \right) \bigg|_{t=0} \]
\[ = \frac{\partial}{\partial t} \int_M \omega^1 \wedge \cdots \wedge \omega^n \bigg|_{t=0} \]
\[ = \int_M \frac{\partial}{\partial t}(\omega^1 \wedge \cdots \wedge \omega^n) \bigg|_{t=0} \]
\[ = n \int_M H(V_N, x)dM. \]
(4.25)
Theorem 4.1. The area variation of a hypersurface $x: M^n \to \mathbb{R}^{n+1}$ depends only on the normal component of the variation vector field. The critical hypersurfaces for the area integral are exactly the hypersurfaces with vanishing mean curvature $H$ defined by (3.7).

Now we calculate the second variation formula of the area integral for a hypersurface $x: M^n \to \mathbb{R}^{n+1}$.

Assume that $x: M^n \to \mathbb{R}^{n+1}$ is a hypersurface with $H \equiv 0$ and $V^T = \sum_i v^i (e_i) \equiv 0$. From (4.21) and (4.23) we have

$$\nu_i^a = \phi_i^a,$$  \hspace{1cm} (4.26)

$$\frac{\partial \omega^j}{\partial t} = \sum_j \left( -\phi^j_i + \nu^i_k \beta A^k_\alpha \phi^\alpha \phi_j^k \right) \omega^j.$$  \hspace{1cm} (4.27)

From (4.14), (4.17) and (4.18) we get

$$d(\omega_{i\alpha} + \phi_{i\alpha} dt) = (\omega^i_j + \phi^i_j dt) \wedge (\omega^\alpha_j + \phi^\alpha_j dt) g^\beta \nu_\alpha = \omega^i_j \wedge \omega^\alpha_j g^\beta \nu_\alpha + \phi^i_j g^\beta \nu_\alpha \omega^j_i \wedge dt + \phi^i_j g^\beta \nu_\alpha \omega^j_i,$$

$$d(\omega_{i\alpha} + \phi_{i\alpha} dt) = \left( dM + dt \wedge \frac{\partial}{\partial t} \right) (\omega_{i\alpha} + \phi_{i\alpha} dt)$$

$$= dM \omega_{i\alpha} + dM \phi_{i\alpha} \wedge dt + dt \wedge \frac{\partial \omega_{i\alpha}}{\partial t}.$$  \hspace{1cm} (4.28)

Then

$$dM \omega_{i\alpha} = \omega^i_j \wedge \omega^j_\alpha.$$

$$\frac{\partial \omega_{i\alpha}}{\partial t} = \frac{\partial}{\partial t} (A^\alpha_i g^\beta_\alpha \omega^j_i + A^\beta_i g^\alpha_\beta \omega^j_i).$$

From

$$\omega_{i\alpha} = \omega^i_j g^\beta_\alpha = A^\beta_i g^\alpha_\beta \omega^j_i$$

we have

$$\frac{\partial \omega_{i\alpha}}{\partial t} = \frac{\partial}{\partial t} (A^\beta_i g^\alpha_\beta) \omega^j_i + A^\beta_i g^\alpha_\beta \frac{\partial \omega^j_i}{\partial t}.$$  \hspace{1cm} (4.29)

Then

$$\frac{\partial}{\partial t} (A^\beta_i g^\alpha_\beta) \omega^j_i = \frac{\partial \omega_{i\alpha}}{\partial t} - A^\beta_i g^\alpha_\beta \frac{\partial \omega^j_i}{\partial t}$$

$$= (\phi_{i\alpha,k} + \phi^i_j A^\beta_k g^\alpha_\beta) \omega^j_i - A^\beta_i g^\alpha_\beta (-\phi^j_i + v^\alpha \gamma^\alpha \gamma^j_\alpha) \omega^j_k$$

$$= (\phi_{i\alpha,k} + \phi^i_j A^\beta_k g^\alpha_\beta + A^\beta_i g^\beta_\alpha \phi^\alpha_k - A^\beta_i g^\beta_\alpha v^\alpha \gamma^\alpha \gamma^j_i) \omega^j_k.$$  \hspace{1cm} (4.30)

Therefore, with (4.26) we obtain

$$\frac{\partial}{\partial t} (A^\beta_i g^\alpha_\beta) = \nu_{i\alpha} + \phi^i_j A^\beta_k g^\alpha_\beta + A^\beta_i g^\beta_\alpha \phi^\alpha_k - A^\beta_i g^\beta_\alpha v^\alpha \gamma^\alpha \gamma^j_i.$$
Since $\phi_{ij} = -\phi_{ji}$, by (4.30) we get
\[
\sum_i \frac{\partial}{\partial t} (A^\beta_{ii} g_{\beta\alpha}) = \sum_i v_{\alpha,ii} - \sum_i A^\beta_{ik} g_{\beta\alpha} v^e A^\gamma_{ii} g_{\tau\gamma} g^{k\ell}. \tag{4.31}
\]

**Theorem 4.2.** Let $x : M^n \to \mathbb{Q}^{n+1} \subset \mathbb{E}^n_1$ be a hypersurface with $H \equiv 0$. The second variation formula of the area integral for $x$ is
\[
V''(0) = \frac{d^2}{dt^2} \text{Vol}(M_t) \bigg|_{t=0} = \int_M \frac{\partial}{\partial t} (nH) \bigg|_{t=0} \langle V^N, x \rangle dM = \int_M \{ \langle V^N, x \rangle \Delta \langle V^N, x \rangle - (\langle V^N, x \rangle)^2 \| A \|^2 \} dM. \tag{4.32}
\]
Therefore, the hypersurface $x : M^n \to \mathbb{Q}^{n+1} \subset \mathbb{E}^n_1$ with $H \equiv 0$ satisfies $V''(0) \leq 0$.

**Acknowledgements**

The first author was supported by NSFC, No. 10371013; Joint Research of NSFC and KOSEF, No. 10510637; Northeastern University. The second author was supported by KOSEF, No. R01-2003-000-10004-0; Joint Research of NSFC and KOSEF. The authors thank the referee for the valuable suggestions and the comments.

**References**