Pitch functions of ruled surfaces and B-scrolls in Minkowski 3-space

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In this paper, we define pitch function for any non developable ruled surfaces and use this notion to give a new characterization of B-scrolls in Minkowski 3-space.

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1. Introduction

Graves introduced surfaces in Minkowski 3-space $\mathbb{E}^3_1$ called B-scrolls [1]. A ruled surface with representation $x(u, v) = a(u) + v b(u)$ in Minkowski 3-space $\mathbb{E}^3_1$ is called a B-scroll if $a(u)$ and $b(u)$ satisfy the relations $(a'(u), a'(u)) = (b(u), b(u)) = 0$, $(a'(u), b(u)) = 1$ and

$$
\begin{aligned}
\alpha'(u) &= \lambda(u)\beta(u), \\
\beta'(u) &= -\mu\alpha(u) - \lambda(u)\gamma(u), \\
\gamma'(u) &= \mu\beta(u),
\end{aligned}
$$

where $a'(u) = a(u)$, $\gamma'(u) = b(u)$, $\beta(u) = \gamma(u) \times a(u)$, $(\beta(u), \beta(u)) = 1$, $\mu$ is constant, $(\cdot, \cdot)$ is the inner product and $\times$ the vector product in $\mathbb{E}^3_1$. B-scrolls appear e.g. in many classification results of some special ruled surfaces [2–7]. However, there are very few conclusions one can draw about properties of B-scroll surfaces.

B-scrolls form a much larger class of ruled surfaces in Minkowski 3-space $\mathbb{E}^3_1$, and it is meaningful to study the properties of B-scrolls. In [8,9], the author gives some characterizations and examples of B-scroll surfaces using the cone curve theory given in [10]. In our paper, we give a new characterization of B-scroll using generalized notion of pitch function of ruled surfaces.

The notion of pitch for closed ruled surfaces is defined in [11].

Remark 1.1. Here and in the following, we use $x \cdot y$ to denote the inner product of two vectors $x$ and $y$ in Euclidean 3-space $\mathbb{E}^3$ and $(x, y)$ the inner product of two vectors $x$ and $y$ in Minkowski 3-space $\mathbb{E}^3_1$. 

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Definition 1.1. Let \( x(u, v) = a(u) + vb(u) \) be a non developable ruled surface in Euclidean 3-space \( \mathbb{E}^3 \) with \( b^2(u) = b(u) \cdot b(u) = 1 \). If \( a(u) \) is a closed curve, \( x(u, v) \) is called a closed ruled surface \([11]\).

Definition 1.2. Let \( x(u, v) = a(u) + vb(u) \) be a non developable ruled surface in Euclidean 3-space \( \mathbb{E}^3 \) with \( b^2(u) = 1 \). We call
\[
\delta := -\int_a^b a'(u) \cdot b(u) du
\]
the pitch of the closed ruled surface \( x(u, v) \) \([11]\).

We will generalize this notion of the pitch of the closed ruled surface, called pitch function (or pitch density function), to any non developable ruled surfaces in Euclidean 3-space \( \mathbb{E}^3 \) and Minkowski 3-space \( \mathbb{E}_1^3 \).

2. Ruled surfaces in Euclidean 3-space

Let \( x(u, v) = a(u) + vb(u) \) be a non developable ruled surface in Euclidean 3-space \( \mathbb{E}^3 \) with \( b^2(u) = 1 \) and the parameter \( u \) is the arc length parameter of \( b(u) \) as a unit spherical curve in Euclidean 3-space \( \mathbb{E}^3 \). Furthermore, we assume that the base curve \( a(u) \) of the ruled surface \( x(u, v) \) is the striction line of the surface, that means \( a'(u) \cdot b'(u) = 0 \). Choosing \( \gamma'(u) = b(u) \), \( \gamma'(u) = b(u) \), and \( a(u) = b(u) \times \gamma(u) \), then
\[
\begin{align*}
\beta'(u) &= -\kappa(u)a(u) - \gamma'(u), \\
\gamma'(u) &= \beta(u).
\end{align*}
\]
We call \( \kappa(u) \) the spherical curvature function, \( \{a(u), \beta(u), \gamma(u)\} \) the spherical Frenet frame of (unit) spherical curve \( b(u) \). In this section, we always assume that the surface \( x(u, v) = a(u) + vb(u) \) satisfies these conditions. The orthogonal trajectory of the rulings on \( x(u, v) \) passing through \((u_0, 0)\) is given by
\[
A(u) = a(u) - \left[ \int_{u_0}^u (a'(t) \cdot b(t)) dt \right] b(u).
\]

Remark 2.1. The spherical Frenet frame of a spherical curve defined here is the Frenet frame of the curve in a 2-dimensional surface of Euclidean 3-space \( \mathbb{E}^3 \).

Definition 2.1. The pitch \( \delta(u_0) \) of the ruled surface \( x(u, v) = a(u) + vb(u) \) at \( a(u_0) \) (or \( (u_0, 0) \)) is defined by
\[
\delta(u_0) := \lim_{\Delta u \to 0} \frac{[A(u_0 + \Delta u) - a(u_0 + \Delta u)] \cdot b(u_0 + \Delta u)}{\Delta u} = -a'(u_0) \cdot b(u_0).
\]
We call \( \delta(u) \) the pitch function (or pitch density function) of the ruled surface \( x(u, v) \).

Remark 2.2. From the definition of the pitch function \( \delta(u) \) we know that
\[
\int_{u_1}^{u_2} \delta(u) du = -\int_{u_1}^{u_2} a'(u) \cdot b(u) du
\]
is the signed distance of which the point on the ruled surface \( x(u, v) \) translates along the ruling from \( u_1 \) to \( u_2 \).

The following conclusion shows a characterization of the pitch function of the ruled surface in \( \mathbb{E}^3 \).

Theorem 2.1. The pitch function \( \delta(u) \) of a non developable ruled surface \( x(u, v) = a(u) + vb(u) \) with \( b^2(u) = 1 \) and \( b^2(u) = 1 \) vanishes identity if and only if the surface \( x(u, v) \) is the binormal surface of its striction line.
Proof. It is easy to check that the binormal surface of any curve in \( E^3 \) satisfies \( \delta(u) \equiv 0 \). For the non developable ruled surface \( x(u, v) = a(u) + v b(u) \) with \( b^2(u) = 1, b^2(u) = 1 \) and \( a(u) \) as the striction line of \( x(u, v) \), we denote the spherical Frenet frame of \( b(u) \) with \( \alpha(u), \beta(u), \gamma(u) = b(u) \). If \( \delta(u) = -a'(u) \cdot b(u) \equiv 0 \) we have \( a'(u) \perp b(u) \) and \( a(u) \) is the striction line of \( x(u, v) \) means that \( a'(u) \perp b(u) \). Then we get that \( a'(u) \| (\alpha(u) = \beta(u) \times \gamma(u) = \beta(u) \times b(u)) \). Therefore, it is easy to get that \( (a(u) \times a'(u)) \| (\gamma(u) = b(u)) \), the surface \( x(u, v) \) is the binormal surface of curve \( a(u) \).

**Remark 2.3.** In this case, the parameter \( u \) is the arc length parameter of the curve \( b(u) \), but usually \( u \) is not the arc length parameter of the curve \( a(u) \).

3. Ruled surfaces in Minkowski 3-space

In Minkowski 3-space \( E^3 \), we consider non developable ruled surfaces with spacelike ruling, timelike ruling and lightlike ruling, respectively. We will also use the following notions of the asymptotic orthogonal trajectory of the null curve and the binormal surface of the null curve in \( E^3 \).

**Definition 3.1.** Let \( a(s) \) be a null curve in \( E^3 \) with arc length parameter \( s \) (that means \( \langle a''(s), a''(s) \rangle = 1 \)) and the asymptotic Frenet frame \( \{ \alpha(s) = a(s), \beta(s), \gamma(s) \} \) such that \( \langle \alpha(s), \alpha(s) \rangle = \langle \gamma(s), \gamma(s) \rangle = \langle \alpha(s), \beta(s) \rangle = \langle \beta(s), \gamma(s) \rangle = 0 \). The curve \( b(s) \) in \( E^3 \) is called the asymptotic orthogonal trajectory (or simply, asymptotic trajectory) of the curve \( a(s) \) if \( \langle b(s), \gamma(s) \rangle \equiv 0 \).

**Remark 3.1.** The asymptotic Frenet frame (asymptotic trajectory) defined here is called the Cartan frame in [12].

**Definition 3.2.** Let \( a(s) \) be a null curve in \( E^3 \) with arc length parameter \( s \) and asymptotic Frenet frame \( \{ \alpha(s) = a'(s), \beta(s), \gamma(s) \} \). The ruled surface \( x(s, v) = a(s) + v \gamma(s) \) is called the binormal surface of the null curve \( a(s) \).

3.1. Ruled surfaces with spacelike ruling or timelike ruling

Let \( x(u, v) = a(u) + v b(u) \) be a non developable ruled surface in Minkowski 3-space \( E^3 \) with \( (b(u), b(u)) = \pm 1 \) and \( |\langle b'(u), b'(u) \rangle| = 1 \), that is, \( u \) is the arc length parameter of \( b(u) \) as a curve on de Sitter space \( S^3_1 \) with sectional curvature \( 1 \) (when \( (b(u), b(u)) = 1 \)) or hyperbolic space \( H^3 \) with sectional curvature \( -1 \) (when \( (b(u), b(u)) = -1 \)). We also assume that the base curve \( a(u) \) of the ruled surface \( x(u, v) \) is the striction line of the surface, that is \( a'(u) \cdot b'(u) = 0 \). Choose de Sitter (or hyperbolic) Frenet frame \( \{ \alpha(u), \beta(u), \gamma(u) = b(u) \} \) such that

\[
\begin{aligned}
&b(u) = \gamma(u), \\
&\alpha'(u) = \kappa(u)\beta(u), \\
&\beta'(u) = \kappa(u)\alpha(u) - \varepsilon_0\gamma(u), \\
&\gamma'(u) = \beta(u),
\end{aligned}
\]

for \( (b(u), b(u)) = \varepsilon = 1 \). \( (\beta(u), \beta(u)) = -\langle \alpha(u), \alpha(u) \rangle = \varepsilon_0 = \pm 1 \); or

\[
\begin{aligned}
&b(u) = \gamma(u), \\
&\alpha'(u) = \kappa(u)\beta(u), \\
&\beta'(u) = -\kappa(u)\alpha(u) + \gamma(u), \\
&\gamma'(u) = \beta(u),
\end{aligned}
\]

for \( (b(u), b(u)) = \varepsilon = -1 \). The orthogonal trajectory of the rulings on \( x(u, v) \) passing through \( (u_0, 0) \) is given by

\[
A(u) = a(u) - \varepsilon \left[ \int_{u_0}^{u} \langle a'(t) \cdot b(t) \rangle \, dt \right] b(u).
\]

**Remark 3.2.** The de Sitter (or hyperbolic) Frenet frame of a curve defined here is the Frenet frame of the curve in a 2-dimensional surface of Minkowski 3-space \( E^3_1 \). Formulas (5) and (6) can be written as

\[
\begin{aligned}
&b(u) = \gamma(u), \\
&\alpha'(u) = \kappa(u)\beta(u), \\
&\beta'(u) = \varepsilon\kappa(u)\alpha(u) - \varepsilon_0\varepsilon\gamma(u), \\
&\gamma'(u) = \beta(u),
\end{aligned}
\]

\( \varepsilon = \pm 1, \varepsilon_0 = \pm 1 \) and when \( \varepsilon = -1, \varepsilon_0 = 1 \).
Definition 3.3. The pitch \( \delta(u_0) \) of the ruled surface \( x(u, v) = a(u) + vb(u) \) with spacelike ruling or timelike ruling at \( a(u_0) \) is defined by
\[
\delta(u_0) := \lim_{\Delta u \to 0} \frac{[A(u_0 + \Delta u) - a(u_0 + \Delta u)] \cdot b(u_0 + \Delta u)}{\Delta u} = -a'(u_0) \cdot b(u_0).
\]
We call \( \delta(u) \) the pitch function of the ruled surface \( x(u, v) \) in \( \mathbb{E}^3_1 \).

Theorem 3.1. The pitch function \( \delta(u) \) of a non developable ruled surface \( x(u, v) = a(u) + vb(u) \) with \( \langle b(u), b(u) \rangle = \pm 1 \) and \( |\langle b'(u), b'(u) \rangle| = 1 \) vanishes identity if and only if the surface \( x(u, v) \) is the binormal surface of its striction line.

Proof. It is easy to check that the binormal surface of any spacelike (with non null normal vector) or timelike curve in \( \mathbb{E}^3_1 \) satisfies \( \delta(u) \equiv 0 \). For the non developable ruled surface \( x(u, v) = a(u) + vb(u) \) with \( \langle b(u), b(u) \rangle = \pm 1 \) and \( \langle b'(u), b'(u) \rangle = 1 \) and \( a(u) \) as the striction line of the surface, we denote the de Sitter (or hyperbolic) Frenet frame of \( b(u) \) with \( \langle \alpha(u), \beta(u), \gamma(u) = b(u) \rangle \). If \( \delta(u) = -a'(u) \cdot b(u) \equiv 0 \) we have \( a'(u) \parallel b(u) \) and \( a(u) \) is the striction line of the surface \( x(u, v) \) means that \( a'(u) \perp b(u) \). Then we get that \( a'(u) \parallel a(u) \parallel (\beta(u) \times \gamma(u) = \beta(u) \times b(u)) \). Therefore, the surface \( x(u, v) \) is the binormal surface of (spacelike or timelike) curve \( a(u) \).

Remark 3.3. For the ruled surface \( x(u, v) = a(u) + vb(u) \) in \( \mathbb{E}^3_1 \) with \( \langle b(u), b(u) \rangle = 0 \) and \( \langle b'(u), b'(u) \rangle = 0 \), we have
\[
\langle \alpha(u), \beta(u) \rangle = \langle \beta(u), \gamma(u) \rangle = \langle \alpha(u), \gamma(u) \rangle = 0, \langle \alpha'(u), \beta(u) \rangle = \langle \gamma(u), \gamma(u) \rangle = 1
\]
and
\[
\begin{align*}
\langle b(u), \gamma(u) \rangle &= \gamma(u), \\
\langle \alpha'(u), \beta(u) \rangle &= -\kappa(u)\alpha(u) - \gamma(u), \\
\langle \beta'(u), b(u) \rangle &= \kappa(u)\beta(u), \\
\gamma'(u) &= \beta(u).
\end{align*}
\]
We can define
\[
\delta(u) du = -d\alpha(u) \cdot b(u),
\]
called pitch form (or pitch density form). If \( \delta(u) \equiv 0 \) we have \( a'(u) \perp b(u) \). Therefore \( a(u) \) is a special null curve if \( a(u) \) is also the striction line of the surface. The ruled surface \( x(u, v) \) is the normal surface of a special null curve \( a(u) \) (generalized null cubic).

3.2. Ruled surfaces with lightlike ruling

Let \( x(u, v) = a(u) + vb(u) \) be a non developable ruled surface in Minkowski 3-space \( \mathbb{E}^3_1 \) with lightlike ruling \( b(u) \) and \( u \) be the arc length parameter of \( b(u) \) as a curve on the lightlike cone \( Q^2 \subset \mathbb{E}^3_1 \). We assume that the base curve \( a(u) \) of the ruled surface \( x(u, v) \) is the striction line of the surface, that is \( a'(u) \cdot b(u) = 0 \). Choose the cone Frenet frame \( \{ \alpha(u), \beta(u), \gamma(u) = b(u) \} \) such that [10]
\[
\begin{align*}
\langle b(u), \gamma(u) \rangle &= \gamma(u), \\
\langle \alpha'(u), \beta(u) \rangle &= \kappa(u)\beta(u), \\
\langle \beta'(u), b(u) \rangle &= \alpha(u) - \kappa(u)\gamma(u), \\
\gamma'(u) &= \beta(u).
\end{align*}
\]
The asymptotic orthogonal trajectory of the lightlike rulings on \( x(u, v) \) passing through \( (u_0, 0) \) is given by
\[
A(u) = a(u) - \left[ \int_{u_0}^u (a'(t) \cdot \alpha(t)) \, dt \right] b(u).
\]

Definition 3.4. The pitch \( \delta(u_0) \) of the ruled surface \( x(u, v) = a(u) + vb(u) \) with lightlike ruling \( b(u) \) and cone Frenet frame \( \{ \alpha(u), \beta(u), \gamma(u) = b(u) \} \) at \( a(u_0) \) is defined by
\[
\delta(u_0) := \lim_{\Delta u \to 0} \frac{[A(u_0 + \Delta u) - a(u_0 + \Delta u)] \cdot \alpha(u_0 + \Delta u)}{\Delta u} = -a'(u_0) \cdot \alpha(u_0).
\]
We call \( \delta(u) \) the pitch function of the ruled surface \( x(u, v) \).

Theorem 3.2. The pitch function \( \delta(u) \) of a non developable ruled surface \( x(u, v) = a(u) + vb(u) \) with \( \langle b(u), b(u) \rangle = 0 \) and \( \langle b'(u), b'(u) \rangle = 1 \) vanishes identity if and only if the surface \( x(u, v) \) is the binormal surface of its striction line.

Proof. It is easy to check that the binormal surface of any null curve with arc length parameter satisfies \( \delta(u) \equiv 0 \). For the non developable ruled surface \( x(u, v) = a(u) + vb(u) \) with \( \langle b(u), b(u) \rangle = 0, \langle b'(u), b'(u) \rangle = 1 \) and \( a(u) \) as the striction line...
of \( x(u, v) \), we denote the cone Frenet frame of \( b(u) \) with \( \{ \alpha(u), \beta(u), \gamma(u) = b(u) \} \). If \( \delta(u) = -a'(u) \cdot \alpha(u) \equiv 0 \) we have \( a'(u) \perp \alpha(u) \) or \( a'(u) \parallel \alpha(u) \) since \( \alpha(u) \) is lightlike. But \( a(u) \) is the striction line of \( x(u, v) \) means that \( a'(u) \perp \beta(u) \). So we get that \( a'(u) \parallel \alpha(u) \). Therefore, the surface \( x(u, v) \) is the binormal surface of null curve \( a(u) \). □

**Theorem 3.3.** The non developable ruled surface in Minkowski 3-space \( \mathbb{E}^3_1 \) with lightlike ruling is a B-scroll if and only if its pitch function vanishes identity.

**Proof.** From Theorem 2 of [9] we know that the binormal ruled surface of a null curve with arc length parameter in \( \mathbb{E}^3_1 \) is a B-scroll. Then with Theorem 3.2 we get the conclusion of Theorem 3.3. □

**Theorem 3.4.** For any curve \( a(u) \), cone curve \( b(u) \) and parametrization \( \{ u, v \} \), the non developable ruled surface \( x(u, v) = a(u) + v b(u) \) in Minkowski 3-space \( \mathbb{E}^3_1 \) is a B-scroll if

\[
\begin{align*}
\langle b(u), b'(u) \rangle &= \mathbf{B}(u), \\
0 &= \langle a'(u), a'(u) \rangle - 2 \langle a'(u), B(u) \rangle \langle a'(u), B'(u) \rangle' - \langle a'(u), B'(u) \rangle^2.
\end{align*}
\]

**Proof.** For any curve \( a(u) \in \mathbb{E}^3_1 \) and cone curve \( b(u) \in \mathbb{Q}^2 \in \mathbb{E}^3_1 \), the surface \( x(u, v) = a(u) + v b(u) \) is a ruled surface with lightlike ruling \( b(u) \). Assume that \( x(u, v) \) is non developable, that means \( b'(u) \neq 0, a'(u) \neq 0 \) and \( a'(u) \not\parallel b(u) \). Putting

\[ b(u) = \sqrt{\langle b(u), b'(u) \rangle} B(u) \]

and by a parameter transformation of \( v \) we know that the parameter \( u \) is the arc length parameter of cone curve \( B(u) \) of the ruling of the surface \( x(u, v) = a(u) + v B(u) \). The striction line of \( x(u, v) = a(u) + v B(u) \) is

\[ A(u) = a(u) - \langle a'(u), B'(u) \rangle B(u). \]

From Theorems 3.2 and 3.3 (or Theorem 3 in [8] or Proposition 2 in [9]) we know that \( x(u, v) = a(u) + v B(u) \) is a B-scroll if and only if \( A(u) \) is a null curve. Therefore we get

\[ 0 = \langle A'(u), A'(u) \rangle = \langle a'(u), a'(u) \rangle - 2 \langle a'(u), B(u) \rangle \langle a'(u), B'(u) \rangle' - \langle a'(u), B'(u) \rangle^2. \] □

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**Appendix**

**Definition A.1.** Let \( a(s) \) be a space curve in Euclidean 3-space \( \mathbb{E}^3 \) with the arc length parameter \( s \) and the Frenet frame \( \{ \alpha(s) = \frac{d\alpha(s)}{ds}, \beta(s), \gamma(s) \} \), that is

\[
\begin{align*}
\mathbf{\hat{a}}(s) &= \alpha(s), \\
\alpha(s) &= \kappa(s) \beta(s), \\
\beta(s) &= -\kappa(s) \alpha(s) + \tau(s) \gamma(s), \\
\gamma(s) &= -\tau(s) \beta(s).
\end{align*}
\]

The ruled surface \( x(s, v) = a(s) + v \gamma(s) \) is called the binormal surface of the curve \( a(s) \) in \( \mathbb{E}^3 \).

**Definition A.2.** Let \( a(s) \) be a spacelike space curve in \( \mathbb{E}^3_1 \) with the arc length parameter \( s \) and the Frenet frame \( \{ \alpha(s) = \frac{d\alpha(s)}{ds}, \beta(s), \gamma(s) \} \), that is

\[
\begin{align*}
\mathbf{\hat{a}}(s) &= \alpha(s), \\
\alpha(s) &= \kappa(s) \beta(s), \\
\beta(s) &= -\varepsilon \kappa(s) \alpha(s) + \tau(s) \gamma(s), \\
\gamma(s) &= \varepsilon \tau(s) \beta(s),
\end{align*}
\]

where

\[
\begin{align*}
\langle \alpha(s), \alpha(s) \rangle &= 1, \\
\langle \beta(s), \beta(s) \rangle &= \varepsilon = \pm 1, \\
\langle \gamma(s), \gamma(s) \rangle &= -\varepsilon,
\end{align*}
\]

and \( \kappa(s) \geq 0 \). The ruled surface \( x(s, v) = a(s) + v \gamma(s) \) is called the binormal surface of the spacelike curve \( a(s) \) in \( \mathbb{E}^3_1 \).
Definition A.3. Let $a(s)$ be a timelike space curve in Minkowski 3-space $\mathbb{E}^3_1$ with the arc length parameter $s$ and the Frenet frame $\{a(s) = \frac{d}{ds}a(s), \beta(s), \gamma(s)\}$, that is

\[
\begin{align*}
\dot{a}(s) &= a(s), \\
\dot{\alpha}(s) &= \kappa(s)\beta(s), \\
\dot{\beta}(s) &= \kappa(s)a(s) + \tau(s)\gamma(s), \\
\dot{\gamma}(s) &= -\tau(s)\beta(s).
\end{align*}
\]

The ruled surface $x(s, v) = a(s) + v\gamma(s)$ is called the binormal surface of the timelike curve $a(s)$ in $\mathbb{E}^3_1$.

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