Structure and characterization of ruled surfaces in Euclidean 3-space

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ABSTRACT
In this paper, using the elementary method we study ruled surfaces, the simplest foliated submanifolds, in Euclidean 3-space. We define structure functions of the ruled surfaces, the invariants of non-developable ruled surfaces and discuss geometric properties and kinematical characterizations of non-developable ruled surface in Euclidean 3-space.

1. Introduction

Müller [6] introduced the concepts of the pitch and angle of pitch of a closed ruled surface in Euclidean 3-space. In [4,5] the authors generalized these notions to pitch (density) function and angle (density) function of pitch (or according to the kinematical meaning, self spining density function) of any non-developable ruled surfaces in Euclidean 3-space and Minkowski 3-space. Some properties and applications of these notions are also given in [4,5]. For example, the B-scroll in Minkowski 3-space is characterized by that the pitch function of the ruled surface with lightlike ruling vanishes identically (Theorem 3.3 in [5]) (5) for the concept of B-scrolls). In this paper we consider non-developable ruled surfaces in Euclidean 3-space. At first, we give the relations between the structure functions of the ruled surface and the curvature, torsion of the striction line of the ruled surface. Then we study normal ruled surface of the space curves and some properties of the non-developable ruled surfaces using the structure functions. As we know that, the tangent ruled surface of a space curve is developable ruled surface; the binormal ruled surface of a space curve is non pitched ruled surface. Finally, using the theory of limits, we define distance density function, translation density function and unit common perpendicular vector field of the non-developable ruled surface in Euclidean 3-space and give a new kinematical characterization of the non-developable ruled surface and also the relations with the structure functions in Euclidean 3-space. The ruled surfaces are the simplest foliated submanifolds. It is meaningful to generalized our methods to study other foliated submanifolds, for example, the surfaces foliated by circles.

2. Structure functions of ruled surfaces

In this section we recall some concepts and results given in [4,5] (also in [1]). Some of them are closely related to the conclusions given in the next section.

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We denote the Euclidean 3-space by $\mathbb{E}^3$ and a regular parameter surface with the parameters $u$ and $v$ in $\mathbb{E}^3$ by $X(u, v)$.

Let $X(u, v) = a(u) + vb(u)$ be a non-developable ruled surface in $\mathbb{E}^3$ with $b^2(u) = 1$ and the parameter $u$ is the arc length parameter of $b(u)$ as a unit spherical curve in $\mathbb{E}^3$. Furthermore, we assume that the base curve $a(u)$ of the ruled surface $X(u, v)$ is the striction line of the surface, that means $a'(u) \cdot b'(u) = 0$, here we use $a' = \frac{da}{du}$. For convenience we call such expression the standard equation of the non-developable ruled surfaces in Euclidean 3-space. Choosing $X(u) = b(u)$, $x'(u) = a(u)$, and $y(u) = a(u) \times x(u)$, the spherical Frenet formulas of unit spherical curve $b(u)$ can be written as

$$\begin{cases}
x'(u) = a(u), \\
x'(u) = -X(u) + K_g(u)y(u), \\
y'(u) = -K_g(u)x(u).
\end{cases}$$

The function $K_g(u)$ is called the spherical curvature function, $\{x(u), x(u), y(u)\}$ is called the spherical Frenet frame of (unit) spherical curve $b(u)$.

### 2.1. Pitch function of non-developable ruled surfaces

The notion of the pitch function of non-developable ruled surfaces in Euclidean 3-space is defined in [5] as follows. The orthogonal trajectory of the rulings on the ruled surface $X(u, v)$ passing through $(u_0, 0)$ is given by

$$A(u) = a(u) - \int_{u_0}^u (a'(u) \cdot b(u))du b(u).$$

**Definition 2.1.** The pitch \(\delta(u_0)\) of the ruled surface $X(u, v) = a(u) + vb(u)$ at $a(u_0)$ or $(u_0, 0)$ is defined by

$$\delta(u_0) := \lim_{\Delta u \to 0} \frac{[A(u_0 + \Delta u) - a(u_0 + \Delta u)] \cdot b(u_0 + \Delta u)}{\Delta u} = -a'(u_0) \cdot b(u_0).$$

We call \(\delta(u)\) the pitch function (or pitch density function) of the (non-developable) ruled surface $X(u, v)$ [5].

**Remark 2.1.** From the definition of the pitch function \(\delta(u)\) we know that

$$\int_{u_1}^{u_2} \delta(u)du = -\int_{u_1}^{u_2} a'(u) \cdot b(u)du$$

is the signed distance of which the point on the ruled surface $X(u, v)$ translates along the ruling from $u_1$ to $u_2$. Therefore we can call the pitch function of the ruled surface as the translation function of the points on the ruling of the ruled surface [5].

The following conclusion shows a characterization of the pitch function of the ruled surface in $\mathbb{E}^3$ [5].

**Theorem 2.1.** The pitch function $\delta(u)$ of a non-developable ruled surface $X(u, v) = a(u) + vb(u)$ with $b^2(u) = 1$ and $|b'(u)| = 1$ vanishes identically if and only if the surface $X(u, v)$ is the binormal surface of its striction line [5].

### 2.2. Angle function of pitch of non-developable ruled surfaces

The notion of the angle function of pitch of non-developable ruled surfaces in Euclidean 3-space is defined in [4] as follows.

**Definition 2.2.** For one-parameter unit vector field $b(u)$ with $|b'(u)| = 1$, the function

$$\theta(u) := -\langle (b'(u) \times b(u))', b'(u) \rangle$$

is called angle (density) function or called self spinning (density) function of vector field $b(u)$ [4].

**Remark 2.2.** From the definition of the self spinning function of vector field $b(u)$ we know that

$$\theta_0 = \int_{u_1}^{u_2} \theta(u)du$$

is the angle that the vector $b(u)$, as an axis in the direction $b(u)$, rotates from $u_1$ to $u_2$ [4].
Definition 2.3. Let \( X(u, v) = a(u) + v b(u) \) be any non-developable ruled surface in \( \mathbb{R}^3 \) and \( a(u) \) the striction line of \( X(u, v) \) such that \( |b(u)| = |b'(u)| = 1 \). The angle function (self spinning function) of vector field \( b(u) \) is called angle (density) function of pitch or self spinning (density) function of the non-developable ruled surface \( X(u, v) \) [4].

From (2.1) and (2.4) we have
\[
\theta(u) = \kappa_g(u). \tag{2.5}
\]

Proposition 2.1. The angle (density) function of pitch or self spinning (density) function of one-parameter unit vector field \( b(u) \) is the spherical curvature function of which \( b(u) \) is considered as a unit spherical curve in \( \mathbb{R}^3 \) [4].

Remark 2.3. If \( X(u, v) = a(u) + v b(u) \) is a closed ruled surface in \( \mathbb{R}^3 \), then
\[
\oint \theta(u) du = \oint \kappa_g(u) du
\]
is the angle of pitch defined by Müller [6] [4].

2.3. Structure functions of non-developable ruled surfaces

Using the spherical curvature function, pitch function and angle function of pitch, we can define the structure functions and give a classification of non-developable ruled surfaces [4].

Definition 2.4. Let \( X(u, v) = a(u) + v b(u) \) be any non-developable ruled surface and \( a(u) \) the striction line of \( X(u, v) \) such that \( a'(u) = \lambda(u)x(u) + \mu(u)y(u) \), here \( \{x(u), y(u)\} \) is the spherical Frenet frame of the spherical curve \( b(u) \), \( u \) is the arc length parameter of \( b(u) \). Assume that \( \kappa_g(u) \) is the spherical curvature function of \( b(u) \), then \( X(u, v) \) is determined by \( \{\kappa_g(u), \lambda(u), \mu(u)\} \) up to a transformation in Euclidean 3-space. The functions \( \kappa_g(u), \lambda(u) \) and \( \mu(u) \) are called structure functions of the non-developable ruled surface \( X(u, v) \) in Euclidean 3-space [4].

From (2.3) and the definition of the structure function \( \lambda(u) \) we have
\[
\lambda(u) = -\frac{a'}{b'}. \tag{2.6}
\]

Definition 2.5. Let \( X(u, v) = a(u) + v b(u) \) be any non-developable ruled surface in \( \mathbb{R}^3 \) and \( a(u) \) the striction line of \( X(u, v) \) such that \( a'(u) = \lambda(u)x(u) + \mu(u)y(u) \), here \( \{x(u), y(u), z(u)\} \) is the spherical Frenet frame of the spherical curve \( b(u) \), \( u \) is the arc length parameter of \( b(u) \). Then \( X(u, v) \) is called pitched ruled surface if its pitch function \( \lambda(u) \equiv 0 \). Otherwise, \( X(u, v) \) is called non pitched ruled surface [4].

Theorem 2.2. Let \( \{\lambda(u), \mu(u), \kappa_g(u)\} \) be the structure functions of the non-developable ruled surface \( X(u, v) = a(u) + v b(u) \) with \( b'(u) = 1 \) and \( |b'(u)| = 1 \) in \( \mathbb{R}^3 \), then we have the following results [4].

1. If \( X(u, v) \) is non pitched, it is the binormal surface of its striction line and can be written as
\[
X(u, v) = \int_{u_0}^{u} \mu(t)y(t)dt + vb(u).
\]

When the structure function \( \mu(u) \equiv a \) = constant, the surface can be written as
\[
X(u, v) = a \int_{u_0}^{u} y(t)dt + vb(u)
\]
and the surface is a binormal surface of a curve of constant torsion.

2. The pitched ruled surfaces can be written as
\[
X(u, v) = \int_{u_0}^{u} [\lambda(t)x(t) + \mu(t)y(t)]dt + vb(u).
\]

3. Properties of non-developable ruled surface

In this section we consider the properties and relations of the structure functions of non-developable ruled surfaces. We give also the new geometric and kinematical characterizations of the structure functions of the non-developable ruled surfaces.
**Theorem 3.1.** Let \( X(u, v) = a(u) + vb(u) \) be a non-developable ruled surface with \( b^2(u) = 1 \). \( |b'(u)| = 1 \) and \( a(u) \) the striction line of \( X(u, v) \). Then the curvature function \( \kappa(u) \) and torsion function \( \tau(u) \) of the striction line \( a(u) \) of \( X(u, v) \) are given by

\[
\kappa^2 = \frac{(\lambda - \lambda_k \mu)^2(\lambda^2 + \mu^2) + (\lambda' \mu - \lambda \mu')^2}{(\lambda^2 + \mu^2)^3}
\]

and

\[
\tau = \frac{(\lambda - \lambda_k \mu)(\lambda'' \mu - \lambda' \mu') + (\lambda_k \mu + \mu)(\lambda - \lambda_k \mu) + (\lambda' \mu - \lambda \mu')(2\lambda' - 2\lambda_k \mu' - \lambda_k \mu)}{(\lambda - \lambda_k \mu)^2(\lambda^2 + \mu^2) + (\lambda' \mu - \lambda \mu')^2}
\]

**Proof.** From

\[
\begin{align*}
\kappa(u) &= \frac{|a'(u) \times a''(u)|}{|a'(u)|^3}, \\
\tau(u) &= \frac{|a'(u) \times a''(u)|}{|a'(u)|^3},
\end{align*}
\]

by a direct calculation we can get the conclusion of this theorem. \( \square \)

Let \( X(u, v) = a(u) + vb(u) \) be a non-developable ruled surface with the structure functions \( \lambda(u), \mu(u) \) and \( \lambda_k(u) \). The first fundamental quantities of \( X(u, v) \) are

\[
\begin{align*}
E &= \lambda^2(u) + \mu^2(u) + v^2, \\
F &= \lambda(u), \\
G &= 1.
\end{align*}
\]

The unit normal vector is

\[
n = \frac{-\mu(u)\lambda(u) + v\phi(u)}{\sqrt{\mu^2(u) + v^2}}.
\]

The second fundamental quantities are

\[
\begin{align*}
L &= -\frac{\mu(u)\lambda(u) - \mu'(u)v + (\mu(u) + \lambda_k(u)v)}{\sqrt{\mu^2(u) + v^2}}, \\
M &= \frac{-\mu(u)}{\sqrt{\mu^2(u) + v^2}}, \\
N &= 0.
\end{align*}
\]

The Gauss curvature and mean curvature of \( X(u, v) \) are

\[
\begin{align*}
K(u, v) &= \frac{-\mu^2(u)}{(\mu^2(u) + v^2)^2}, \\
H(u, v) &= \frac{\lambda_k(u)v^2 + \mu'(u)v + \lambda_k(u)\mu^2(u) + \lambda(u)\mu(u)}{2\sqrt{(\mu^2(u) + v^2)^3}}.
\end{align*}
\]

**Proposition 3.1.** The Gauss curvature \( K(u, v) \) of a non-developable ruled surface satisfies

\[
16 \left( \frac{1}{K} \frac{\partial K}{\partial v} \right)^2 = K + \frac{\sqrt{-K}}{\mu(u)}
\]

for some one parameter function \( \mu(u) \).

**Proof.** From (3.4) by a direct calculation we can get this conclusion. \( \square \)

**Theorem 3.2.** Let \( a(s) \) be a space curve with the arc length parameter \( s \), curvature function \( \kappa(s) \), torsion function \( \tau(s) \) and Frenet frame \( (\tilde{x}(s), \beta(s), \gamma(s)) \). Then the striction line of the normal surface \( X(u, v) = X(s, v) = a(u) + vb(u) = a(s) + v\beta(s) \) of \( a(s) \) is

\[
A(s) = a(s) + \frac{\kappa(s)}{\kappa^2(s) + \tau^2(s)} \beta(s).
\]
The structure functions of \( X(u, v) \) are
\[
\begin{align*}
\lambda(\tilde{s}) &= \frac{1}{\kappa^2(\tilde{s}) + \tau^2(\tilde{s})} \frac{d}{d\tilde{s}} \left( \frac{\kappa''(\tilde{s})}{\kappa^2(\tilde{s}) + \tau^2(\tilde{s})} \right) = \frac{d}{du} \left( \frac{\kappa''(u)}{\kappa^2(u) + \tau^2(u)} \right), \\
\mu(\tilde{s}) &= \frac{-\tau'(\tilde{s})}{\kappa^2(\tilde{s}) + \tau^2(\tilde{s})^2} = -\frac{\tau(\tilde{s})}{\kappa^2(\tilde{s}) + \tau^2(\tilde{s})^2}, \\
K(\tilde{s}) &= \frac{\kappa'(\tilde{s})^2 + \tau'(\tilde{s})^2}{(\kappa^2(\tilde{s}) + \tau^2(\tilde{s}))^2},
\end{align*}
\] (3.8)

where \( u \) is the arc length parameter of \( \tilde{\beta} \) as a unit spherical curve and
\[
u = \int_{\tilde{s}_0}^\tilde{s} \left[ \kappa(\tilde{s})^2 + \tau(\tilde{s})^2 \right] d\tilde{s}.
\] (3.9)

**Remark 3.1.** For convenience we use here
\[
\begin{align*}
\kappa'(\tilde{s}) &= \frac{d\kappa(\tilde{s})}{d\tilde{s}}, & \kappa'(u) &= \frac{d\kappa(u)}{du}, \\
\tau'(\tilde{s}) &= \frac{d\tau(\tilde{s})}{d\tilde{s}}, & \tau'(u) &= \frac{d\tau(u)}{du}.
\end{align*}
\]

**Proof.** From the definition of the striction line of the ruled surface, by a direct calculation we know that the curve (3.7) is the striction line of \( X(\tilde{s}, v) \).

From \( x = b = \tilde{\beta} \) we have
\[
\begin{align*}
\nu = \vec{\beta} \cdot \frac{d\tilde{\beta}}{du}, \\
\frac{d\tilde{\beta}}{du} &= \frac{1}{\kappa'^2 + \tau'^2}.
\end{align*}
\] (3.10)

Then
\[
y = \nu \times x = -(\tau \nu + \kappa) \frac{d\tilde{s}}{du} = -\Omega \frac{d\tilde{s}}{du},
\]
here \( \Omega \) is the Darboux vector field of the curve \( a(\tilde{s}) \) and we assume that \( \frac{d\tilde{s}}{du} > 0 \). Therefore
\[
\begin{align*}
\lambda &= \left( \frac{d\tilde{\beta}}{du}, \nu \right) = \frac{1}{\kappa'^2 + \tau'^2} \left( -\frac{\kappa''}{\kappa'^2 + \tau'^2} \right)', \\
\mu &= \left( \frac{d\tilde{\beta}}{du}, \nu \right) \cdot \nu = -\frac{\tau''}{\kappa'^2 + \tau'^2}, \\
K &= -\left( \frac{d\tilde{\beta}}{du}, \nu \right) \cdot \nu = \frac{\kappa''(\tilde{s})^2 + \tau''(\tilde{s})^2}{(\kappa'^2 + \tau'^2)^2}.
\end{align*}
\] (3.11)

With (3.10) we get the conclusion of this theorem. \( \square \)

**Corollary 3.3.** The striction line of the normal surface of a Mannheim curve is its Mannheim partner curve.

**Proof.** The curvature function \( \kappa(\tilde{s}) \) and torsion function \( \tau(\tilde{s}) \) of the Mannheim curve satisfy
\[
\kappa = c(\kappa'^2 + \tau'^2)
\]
for some constant \( c \neq 0 \). Therefore from (3.7) we know that the striction line of the normal surface of a Mannheim curve \( a(\tilde{s}) \) is
\[
a(\tilde{s}) + c\tilde{\beta}(\tilde{s}),
\]
which is the Mannheim partner curve of \( a(\tilde{s}) \). \( \square \)

**Remark 3.2.** If there exists a corresponding relation between the space curves \( \Gamma \) and \( \Gamma_1 \) in \( \mathbb{E}^3 \) such that, at the corresponding points of the curves, the normal lines of \( \Gamma \) coincide with the binormal lines of \( \Gamma_1 \), the curve \( \Gamma \) is called a Mannheim curve, and \( \Gamma_1 \) a Mannheim partner curve of \( \Gamma \) \([3]\).

**Proposition 3.2.** Let \( X(u, v) \) be a non-developable ruled surface. If \( X(u, v) \) is a Weingarten surface, the structure functions of \( X(u, v) \) are all constants.
Proof. Let \( X(u, v) = a(u) + vb(u) \) be a non-developable ruled surface with \( b'(u) = 1 \), \( |b'(u)| = 1 \) and \( a(u) \) the striction line of \( X(u, v) \). From (3.4) and (3.5) we have

\[
K_u = \frac{2\mu\nu' (\mu^2 - \nu^2)}{\left(\mu^2 + \nu^2\right)^3},
\]

(3.12)

\[
K_v = \frac{4\mu^2 \nu}{\left(\mu^2 + \nu^2\right)^3},
\]

(3.13)

\[
H_u = \frac{k_g v^3 + \mu v^3 + (2k_g' \mu^2 - \kappa_g \mu \nu') v^2 + \mu(\mu^2 - 3\nu^2) v + \nu^2 (k_g' \mu^2 - \kappa_g \mu \nu' + \nu \nu' - 2\nu \nu')}{2\sqrt{(\mu^2 + \nu^2)^5}},
\]

(3.14)

\[
H_v = \frac{-k_g v^3 - 2\mu v^2 - \mu (\kappa_g + 3\nu) v + \mu^2 \nu'}{2\sqrt{(\mu^2 + \nu^2)^5}}.
\]

(3.15)

Then from \( K_u H_v = K_v H_u \), we have

\[
(2k_g' \mu - \kappa_g \nu') v^3 + 2(\mu^2 \nu - \mu^2) v^3 + \mu(4k_g' \mu^2 - 2\kappa_g \mu^2 \nu' + 2\nu \nu - \nu \nu') v^3 + \mu^2 (2\mu^2 \nu - 3\nu^2) v^3
\]

\[
+ \mu^3 (2k_g' \mu^2 - \kappa_g \mu \nu' + 2\nu \nu - \nu \nu') v - \mu^4 \nu^2 = 0,
\]

that is

\[
\begin{cases}
2k_g' \mu - \kappa_g \nu' = 0, \\
\mu^2 \nu - \mu^2 = 0, \\
4k_g' \mu^2 - 2\kappa_g \mu^2 \nu' + 2\nu \nu - \nu \nu' = 0, \\
2\mu^2 \nu - 3\nu^2 = 0, \\
2k_g' \mu^2 - \kappa_g \mu^2 \nu' + 2\nu \nu - \nu \nu' = 0, \\
\mu^2 \nu^2 = 0.
\end{cases}
\]

(3.17)

Solving these equations we obtain that the structure functions \( \lambda(u) \), \( \mu(u) \) and \( k_g(u) \) of \( X(u, v) \) are all constants. \( \square \)

Remark 3.3. A surface in 3-space is called a Weingarten surface or a W-surface if the two principal curvatures \( k_1 \) and \( k_2 \) are not independent of one another or, equivalently, if a certain relation \( \Phi(k_1, k_2) = 0 \) is identically satisfied on the surface. [2].

In the following we give a kinematical characterization of the structure functions of the ruled surface \( X(u, v) \) in \( \mathbb{E}^3 \).

Definition 3.1. Let \( X(u, v) \) be a non-developable ruled surface in \( \mathbb{E}^3 \). We denote the distance of two rulings \( X(u_0 + \Delta u, v) \) and \( X(u_0, v) \) by \( \text{dist}(u_0, u_0 + \Delta u) \). Then

\[
\text{dist}_X(u_0) := \lim_{\Delta u \to 0} \frac{\text{dist}(u_0, u_0 + \Delta u)}{\Delta u}
\]

(3.18)

is called distance density at \( u_0 \). The function \( \text{dist}_X(u) \) is called distance density function of \( X(u, v) \).

Theorem 3.4. The function \( \mu(u) \) of the structure functions of the non-developable ruled surface \( X(u, v) \) is the density function of the signed distance of the rulings of \( X(u, v) \). That is, the distance of two rulings \( X(u_1, v) \) and \( X(u_2, v) \) is given by

\[
\text{dist}(u_1, u_2) = \pm \int_{u_1}^{u_2} \mu(u) du.
\]

(3.19)

Proof. The distance \( \text{dist}(u_0, u_0 + \Delta u) \) of the rulings \( X(u_0, v) = a(u_0) + vb(u_0) \) and \( X(u_0 + \Delta u, v) \) is given by

\[
\pm \text{dist}(u_0, u_0 + \Delta u) = \frac{(a(u_0 + \Delta u) - a(u_0), b(u_0), b(u_0 + \Delta u))}{|b(u_0) \times b(u_0 + \Delta u)|} = \frac{(a(u_0 + \Delta u) - a(u_0), b(u_0), b(u_0 + \Delta u)) - b(u_0))}{|b(u_0) \times (b(u_0 + \Delta u) - b(u_0))|}.
\]
Therefore
\[
\lim_{\Delta u \to 0} \frac{\pm \text{dist}(u_0, u_0 + \Delta u)}{\Delta u} = \lim_{\Delta u \to 0} \frac{\nabla \text{dist}(u_0, u_0 + \Delta u)}{\Delta u} = \frac{\langle a'(u_0), \overrightarrow{b}(u_0) \rangle}{|\overrightarrow{b}(u_0) \times \overrightarrow{b}'(u_0)|} = \frac{\langle a'(u_0), \overrightarrow{b}(u_0), \overrightarrow{b}'(u_0) \rangle}{|\overrightarrow{b}(u_0) \times \overrightarrow{b}'(u_0)|}.
\]

That is
\[
\text{dist}_X(u) = \pm \mu(u). \quad \Box
\]

**Definition 3.2.** Let \(X(u, v) = a(u) + vb(u)\) be a non-developable ruled surface in \(\mathbb{E}^3\). We denote the translation of the points on the ruling from \(X(u_0, v)\) to \(X(u_0 + \Delta u, v)\) by \(\text{trans}(u_0, u_0 + \Delta u)\). Then
\[
\text{trans}_X(u_0) = \lim_{\Delta u \to 0} \frac{\text{trans}(u_0, u_0 + \Delta u)}{\Delta u} = \pm \lim_{\Delta u \to 0} \frac{\langle a(u_0 + \Delta u) - a(u_0), \overrightarrow{b}(u_0) \rangle}{\Delta u}.
\]

is called translation density at \(u_0\). The function \(\text{trans}_X(u)\) is called translation density function of \(X(u, v)\).

**Theorem 3.5.** The function \(\lambda(u)\) of the structure functions of the non-developable ruled surface \(X(u, v)\) is the density function of the signed translation of the rulings of \(X(u, v)\). That is, the translation distance of the points on the ruling from \(X(u_1, v)\) to \(X(u_2, v)\) is given by
\[
\text{trans}(u_1, u_2) = \pm \int_{u_1}^{u_2} \lambda(u) \, du.
\]

**Definition 3.3.** Let \(X(u, v) = a(u) + vb(u)\) be a non-developable ruled surface in \(\mathbb{E}^3\). For two rulings at \(X(u_0, v)\) and \(X(u_0 + \Delta u, v)\) we define
\[
\text{normal}_X(u_0) = \lim_{\Delta u \to 0} \frac{b(u_0) \times b(u_0 + \Delta u)}{|b(u_0) \times b(u_0 + \Delta u)|} = \lim_{\Delta u \to 0} \frac{b(u_0) \times b(u_0 + \Delta u)}{|b(u_0 + \Delta u) - b(u_0)|} = \frac{b(u_0) \times b'(u_0)}{|b(u_0) \times b'(u_0)|}.
\]

The vector \(\text{normal}_X(u_0)\) is called unit common perpendicular vector of the ruling at \(u_0\). The vector field \(\text{normal}_X(u)\) is called unit common perpendicular vector field of the ruling of \(X(u, v)\).

**Theorem 3.6.** The function \(\kappa_s(u)\) of the structure functions of the non-developable ruled surface \(X(u, v)\) is signed spinning speed function of the unit common perpendicular vector field of the ruling of \(X(u, v)\).

**Proof.** From (2.1), (3.22) and the definition of the spinning speed function of the unit vector field we easily get the conclusion of this theorem. \(\Box\)

**Remark 3.4.** It is well known that any space curve is the striction line of its tangent surface and also the striction line of its binormal surface. And it is easy to prove that the striction line of the (principal) normal surface of any space curve \(\Gamma_0\) is always a new space curve \(\Gamma_1\) than \(\Gamma_0\). Therefore, in this way, considering the striction line of the normal surface of space curve we can get a sequence of curves, \(\Gamma_0, \Gamma_1, \Gamma_2, \ldots, \Gamma_n\). The properties and the relationship between these curves and their normal surfaces are very interesting problems. For example, it is easy to see that when \(\Gamma_0\) is a Mannheim curve, \(\Gamma_1\) is the partner curve of \(\Gamma_0\). Using the structure functions of ruled surface we can give the necessary and sufficient conditions that a ruled surface is the normal surface of a space curve. The examples and applications of the results in this paper can be found also in the following problems.

- Consider the classification of ruled surfaces with constant \(\kappa_s, \mu\) or \(\lambda\).
- Study the surfaces foliated by circles or other special planar curves using the methods of structure functions. In a general way, study line congruence and motion of one parameter plan.
- Study ruled submanifolds and foliation structures.

**References**