Hopf bifurcation in an Internet worm propagation model with time delay in quarantine

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Internet worm attacks reduce network security and cause economic losses. The use of a quarantine strategy is prominent in defending against worms, and it has been applied to various worm propagation models. Although theoretical analysis suggests that worms must get eliminated under quarantine, such a result does not appear in a real network. The time delay considered in this paper, which is caused by the time window of the intrusion detection system (IDS) that exists in the propagation system, is one of the main reasons for this. A worm propagation model with time delay under quarantine is constructed for practical application. The stability of the positive equilibrium and local Hopf bifurcation are discussed. By analysis, a critical value $\tau_0$ of the Hopf bifurcation is derived. When the time delay is less than $\tau_0$, the worm propagation system is stable and easy to predict; when it is equal to or greater than $\tau_0$, Hopf bifurcation appears. Since it is easy to control and eliminate worms under a simple and stable worm propagation system without Hopf bifurcation, the time window of the IDS must be adjusted so that the time delay is less than $\tau_0$, which ensures that the worm propagation system remains stable and that worms can be eliminated with certain containment strategy. Numerical results from our experiment support our theoretical analysis.

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1. Introduction

Many models have been constructed to predict the spread of worms, and some worm prevention strategies have been taken into consideration. The SIR (susceptible, infected, and removed) model, which borrows from biological epidemiology, plays a significant role in the research field of worm propagation modeling [1]. In spite of considering certain human factors, the model cannot perform the containment of worms effectively.

The use of a quarantine strategy has made great contributions to disease control, and it has been thus adapted to defend a system against worms. In the computer field, the implementation of quarantine measures relies on an intrusion detection system (IDS) [2]. The intrusion detection system has two parts: a misuse intrusion detection system and an anomaly intrusion detection system. The anomaly detection system is commonly used to detect malicious code such as computer viruses and worms, to ensure relatively better performance [3,4]. In such a system, the normal behavior database is built in advance. Once a deviation from the normal behavior is detected, such behavior is recognized as an attack, and appropriate response actions, such as vaccination and quarantine, are triggered.

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However, anomaly-based detection approaches suffer from the disadvantage of false alarms. To make a trade off between the false negative rate and false positive rate, the mechanism of time windows is thus adapted in the IDS. That is, the decision of whether an alarm is a true or false positive is based on the number of abnormal behaviors detected in a time window. This means that the size of the time window affects both the true and false positive rates. However, the setting of time windows can lead to time delay. Therefore, under quarantine defense based on an IDS, the time delay should be considered to accord with actual conditions. In this paper, a time delay is introduced in a worm propagation model with quarantine, the stability of which is analyzed. Through analysis, it is found that the introduction of a time delay increases the complexity of worm propagation. Too large a time delay may lead to Hopf bifurcation, and a worm propagation system with Hopf bifurcation could not help to eliminate worms. This implies that the time delay should be decreased properly through a decrease of the window size in order to guarantee simplification and stability of the worm propagation system. In a real network environment, of course, many other factors besides time delay can influence a worm propagation system. For instance, in [5], the authors study the effect of network congestion on worm propagation; however, in this model, the transition delays are captured by exclusively exponentially distributed events. By contrast, in this paper, we consider incorporating a deterministic time delay, which has a significant effect on the dynamics of the system. This paper thus focuses on the effect of time delay on a worm propagation system, and other factors will be taken into consideration in our future work.

The structure of this paper is as follows. In Section 2, related work on the time delay model is provided. Section 3 gives a brief introduction of the SIDQV model and analysis of the stability of equilibrium. We present our time delay model and analyze its threshold of Hopf bifurcation. Then in Section 4, we present numerical results for two worms. Finally, Section 5 gives the conclusions of the paper.

2. Related works

The model of worm propagation helps us acknowledge the characteristics of worms and predict the trend of worm propagation. The Kermack–McKendrick model, also called the SIR model, is introduced into the research on worms. Qing analyzes and simulates the SIR model in [1]. Though some human factors are included, this model cannot restrain worms effectively. Thus, a variety of containment strategies have emerged, and they have been applied to worm propagation models.

The use of a quarantine strategy has produced a tremendous effect on controlling disease. Inspired by this, quarantine is also widely used in worm containment [3,4,6,7]. Toutonji presents the PWDQ (passive worm dynamic quarantine) worm propagation model, combining passive benign worms and a dynamic quarantine strategy [3]. In theory, the combination of two kinds of containment technique can slow down the speed of spread of worms. Based on the two-factor model, Zou proposes a worm propagation model under dynamic quarantine defense [4]. In the model, “assume guilty before proven innocent” is the fundamental principle, and the users’ experience cannot be weakened as possible. Worms can be restrained without affecting users’ getting online. Wang puts forward the SEIQV (susceptible, exposed, infected, quarantined, and vaccinated) model, which combines vaccination and a dynamic quarantine strategy [6]. As we know, vaccination has gained prominence in controlling the spread of disease, and it has also been applied to network worms [8,9]. However, there is a time delay in the actual network and these models with worm containment strategies do not consider this problem.

Some scholars have done research on the time delay. Since there are distinct characteristics at different periods of worm propagation, a section model of worm propagation based on the two-factor model has been constructed by Wang [10]. A delay is added into the model for the reason that reassembling a system or killing a virus requires some time. The stability of the positive equilibrium of this model is discussed and the existence of Hopf bifurcation of the model is proved. With a delayed eco-epidemiological system, Wang considers the delay as the bifurcation parameter and analyzes the characteristic equation of the system [11]. The stability of the positive equilibrium and the existence of Hopf bifurcation are studied. Both papers pave the way for studying a worm propagation model with time delay and Hopf bifurcations.

3. Worm propagation model and its stability analysis

3.1. The simple epidemic model

In the simple epidemic model (SEM), every host will stay in one of two states: susceptible or infected. At the beginning of a worm outbreak, all hosts in the Internet are susceptible and vulnerable. The hosts will be infected entirely in a short time. The transition diagram is given in Fig. 1.

![State Transition Diagram of the SEM](https://via.placeholder.com/150)

Fig. 1. The state transition diagram of the SEM.
Table 1
Notation used in this paper.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N )</td>
<td>Total number of hosts under consideration</td>
</tr>
<tr>
<td>( S(t) )</td>
<td>Number of susceptible hosts at time ( t )</td>
</tr>
<tr>
<td>( I(t) )</td>
<td>Number of infected hosts at time ( t )</td>
</tr>
<tr>
<td>( D(t) )</td>
<td>Number of quarantined infected and susceptible hosts at time ( t )</td>
</tr>
<tr>
<td>( Q(t) )</td>
<td>Number of quarantined infected and susceptible hosts at time ( t - \tau )</td>
</tr>
<tr>
<td>( V(t) )</td>
<td>Number of immunization hosts at time ( t )</td>
</tr>
<tr>
<td>( \alpha_1 )</td>
<td>Quarantine probability of susceptible hosts at time ( t )</td>
</tr>
<tr>
<td>( \alpha_2 )</td>
<td>Quarantine probability of infected hosts at time ( t )</td>
</tr>
<tr>
<td>( \beta )</td>
<td>Infected rate at time ( t ) in the worm propagation model</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>Removal rate of infected hosts at time ( t )</td>
</tr>
<tr>
<td>( \mu )</td>
<td>Reassembly rate of immunization hosts at time ( t )</td>
</tr>
<tr>
<td>( \delta )</td>
<td>Removal rate of quarantine hosts at time ( t )</td>
</tr>
<tr>
<td>( \tau )</td>
<td>The period of time delay of detecting by the anomaly detection system</td>
</tr>
</tbody>
</table>

From the state transition diagram, the differential equations of the SEM are straightforward to formulate:

\[
\begin{align*}
\frac{dS}{dt} &= -\beta SI \\
\frac{dI}{dt} &= \beta SI - \gamma I \\
\frac{dV}{dt} &= \gamma I - \mu V.
\end{align*}
\]  (3.1)

In the model, it can be supposed that \( N \) is the total number of hosts in the Internet; \( S(t) \) is defined as the proportion of susceptible hosts at time \( t \); \( I(t) \) is defined as the proportion of infected hosts at time \( t \); \( \beta > 0 \) is the infection rate. The notation used in the paper is shown in Table 1.

It is obvious that system (3.1) has a unique globally asymptotically stable equilibrium \((0, N)\).

As mentioned above, it can be obtained that \( I(t) \) must be infinitely close to \( N \). But in fact hardly any kind of worm can infect all hosts on the Internet.

3.2. The SIVS model

The model consists of three states: susceptible, infected and vaccinated. Due to no permanent immunization, vaccinated hosts can become susceptible ones. Some appropriate assumptions are given as follows: \( V(t) \) is defined as the proportion of vaccinated hosts at time \( t \); the rate \( \gamma \) of immunization from \( I(t) \) to \( V(t) \) is greater than zero because worms can be artificially removed by using anti-virus software; \( \mu \) is the rate of vaccination failure because of the reassembly of the computer operating system. The transition diagram of the quarantine model with newborn is given in Fig. 2.

From the state transition diagram, the differential equations of the SIVS model are formulated as follows:

\[
\begin{align*}
\frac{dS}{dt} &= -\beta SI + \mu V \\
\frac{dI}{dt} &= \beta SI - \gamma I \\
\frac{dV}{dt} &= \gamma I - \mu V.
\end{align*}
\]  (3.2)

**Theorem 3.1.** Take system (3.2).

(a) If \( R_0 < 1 \), system (3.2) has a unique globally asymptotically stable equilibrium point \( E_0 \) \((N, 0, 0)\).

(b) If \( R_0 > 1 \), system (3.2) has two equilibrium points. The equilibrium point \( E_0 \) \((N, 0, 0)\) is unstable, and the equilibrium \( E_1 \) \((S_1, I_1, V_1)\) is locally asymptotically stable.

Here, \( R_0 = \frac{\beta N}{\gamma} \), and \( S_1 = \frac{\gamma}{\beta} \cdot I_1 = \frac{N\beta - \gamma}{\mu + \gamma} \cdot \frac{\mu}{\beta} \), \( V_1 = \frac{\gamma I_1}{\mu} \).

The proof can be found in [12].
3.3. The SIDQV model

This model includes five states: susceptible, infected, delayed, quarantined, vaccinated. Here the intrusion detection system is corporated and hosts detected by such system will be quarantined. Since time delay exists, a ‘delayed’ state is added. Quarantine is an effective measure to defend against disease in the real world. Hence, quarantine is also introduced to constrain the spread of worms in order to make up for there being no timely patching hosts. An abnormal host can be detected by the IDS when a port or several ports of it generate illegal scans. With the help of switches or routers, such a host can be quarantined in a way that only the port or these several ports of illegal scans can be blocked and other ports are used normally, which can minimize the effects on users’ getting online [7]. Then, these quarantined hosts can be patched and released.

The implementation of a quarantine strategy relies on the IDS. In this study, the anomaly intrusion detections system is adapted for its common use. The false alarm of the anomaly IDS thus has to be considered, and the mechanism of time windows can help solve the problem. Whether an alarm is raised or not depends on the number of abnormal behaviors detected in the particular size of time window chosen [4]. If the cumulative number of abnormal behaviors reaches the threshold set in the IDS in advance, an alarm is raised to indicate that the current behavior is judged as an attack, and certain defensive measure will be taken. If the window size increases, the accumulated number of normal behaviors also increases during this time, and the threshold should thus be raised. The larger the window size is, the lower the false positive rate is because enough time is used to judge if such behavior is occasional or not. Therefore, the false positive rate of the anomaly detection system can be reduced by increasing its window size. In an actual environment, to reduce the number of false alarms, an attack can be carefully identified by the anomaly detection system in a relative large window size. However, the mechanism of time windows also leads to a time delay.

According to the two worm propagation models above, some appropriate assumptions are given as follows: \( D(t) \) is defined as the proportion of susceptible hosts and infected hosts to be quarantined at time \( t \); \( \alpha_1 \), which denotes the rate of quarantine from \( S(t) \) to \( D(t) \), is greater than zero, because the system quarantines worms by the anomaly detection system; \( \alpha_2 \), which denotes the rate of quarantine from \( I(t) \) to \( D(t) \), is greater than zero, because the system quarantines worms by the anomaly detection system; \( Q(t) \) is defined as the proportion of quarantine at time \( t - \tau \); \( \delta \), which denotes the rate of immunization from \( Q(t) \) to \( V(t) \), is greater than zero, because the anti-virus system is killing worms; \( \tau \), which denotes the period of time delay of detection by the anomaly detection system, is greater than zero. The transition diagram is given in Fig. 3.

From the state transition diagram, the differential equations of the SIDRV model are formulated as follows:

\[
\begin{align*}
\frac{dS}{dt} &= -\beta SI - \alpha_1 S + \mu V \\
\frac{dI}{dt} &= \beta SI - \gamma I - \alpha_2 I \\
\frac{dD}{dt} &= \alpha_1 S(t) - \alpha_1 S(t - \tau) + \alpha_2 I(t) - \alpha_2 I(t - \tau) \\
\frac{dQ}{dt} &= \alpha_1 S(t - \tau) + \alpha_2 I(t - \tau) - \delta Q \\
\frac{dV}{dt} &= \gamma I + \delta Q - \mu V.
\end{align*}
\]

(3.3)

The population size is set as \( N \), which is normalized to unity:

\[ S(t) + I(t) + D(t) + Q(t) + V(t) = N. \]

(3.4)
3.4. Stability of the positive equilibrium and local Hopf bifurcations

It is easy to see that system (3.3) has a unique positive equilibrium \( E^* (S^*, I^*, D^*, Q^*, V^*) \), provided that the condition

\[
(H_1) \quad \frac{N}{\gamma + \alpha_2} > 1
\]

is satisfied, where

\[
S^* = \frac{\gamma + \alpha_2}{\beta}, \quad D^* = 0, \quad Q^* = \frac{\alpha_1 S^* + \alpha_2 I^*}{\delta}, \quad V^* = \frac{\beta S^* I^* + \alpha_1 S^*}{\mu}.
\]

**Proof.** From system (3.3), we can get

\[
\begin{align*}
S &= \frac{\gamma + \alpha_2}{\beta} \\
D &= 0 \\
Q &= \alpha_1 S^* + \alpha_2 I^* \\
V &= \frac{\beta S^* I^* + \alpha_1 S^*}{\mu}.
\end{align*}
\] (3.5)

Since \( S + I + D + Q + V = N \),

\[
S^* + \frac{\alpha_1 S^* + \alpha_2 I^*}{\delta} + I^* + \frac{\beta S^* I^* + \alpha_1 S^*}{\mu} = N. \tag{3.6}
\]

If \((H_1)\) is satisfied, Eq. (3.6) has one unique positive root, \( I^* \). So there is one unique positive equilibrium \( E^* (S^*, I^*, D^*, Q^*, V^*) \) of system (3.3). The proof is completed. \( \Box \)

Since \( S + I + D + Q + V = N \), \( V = N - S - I - D - Q \); system (3.3) is equivalent to the differential system (3.7):

\[
\begin{align*}
\frac{dS}{dt} &= -\beta SI - \alpha_1 S + \mu(N - S - I - D - Q) \\
\frac{dI}{dt} &= \beta SI - \gamma I - \alpha_2 I \\
\frac{dD}{dt} &= \alpha_1 S(t) - \alpha_1 S(t - \tau) + \alpha_2 I(t) - \alpha_2 I(t - \tau) \\
\frac{dQ}{dt} &= \alpha_1 S(t - \tau) + \alpha_2 I(t - \tau) - \delta Q.
\end{align*}
\] (3.7)

By substituting \( S^*, I^*, D^* \) and \( Q^* \) into \( S, I, D \) and \( Q \) respectively, the Jacobi matrix of system (3.7) about is given by

\[
J(E^*) = \begin{pmatrix}
\lambda + A_1 & A_2 & \mu & \mu \\
-\beta I^* & \lambda & 0 & 0 \\
-\alpha_1 + \alpha_1 e^{-\lambda \tau} & -\alpha_2 + \alpha_2 e^{-\lambda \tau} & \lambda & 0 \\
-\alpha_1 e^{-\lambda \tau} & -\alpha_2 e^{-\lambda \tau} & 0 & \lambda + \delta
\end{pmatrix},
\] (3.8)

where

\[
A_1 = \beta I^* + \alpha_1 + \mu > 0, \quad A_2 = \beta S^* + \mu = \alpha_2 + \gamma + \mu > 0,
\]

and thus the characteristic equation of system (3.7) is given by

\[
P(\lambda) + Q(\lambda) e^{-\lambda \tau} = 0, \tag{3.9}
\]

where

\[
P(\lambda) = \lambda^4 + (A_1 + \delta) \lambda^3 + (\alpha_1 \mu + A_1 \delta - \beta I^* A_2) \lambda^2 + (\beta I^* \alpha_2 \mu - \delta \beta I^* A_2 + \alpha_1 \mu \delta) \lambda + \mu \beta I^* \alpha_2 \delta,
\]

\[
Q(\lambda) = -\delta \alpha_1 \mu \lambda - \delta \beta I^* \alpha_2 \mu.
\]

Let

\[
p_3 = A_1 + \delta, \quad p_2 = \alpha_1 \mu + A_1 \delta - \beta I^* A_2, \quad p_1 = \beta I^* \alpha_2 \mu - \delta \beta I^* A_2 + \alpha_1 \mu \delta, \quad p_0 = \beta I^* \alpha_2 \delta,
\]

\[
q_1 = -\delta \alpha_1 \mu, \quad q_0 = -\delta \beta I^* \alpha_2 \mu.
\]

Then

\[
P(\lambda) = \lambda^4 + p_3 \lambda^3 + p_2 \lambda^2 + p_1 \lambda + p_0, \quad Q(\lambda) = q_1 \lambda + q_0.
\]
When \( \tau = 0 \), Eq. (3.9) reduces to
\[
\lambda^4 + p_3\lambda^3 + p_2\lambda^2 + (p_1 + q_1)\lambda + (p_0 + q_0) = 0.
\] (3.10)

Suppose that the following holds:
\[\text{(H}_2\text{)} \quad p_3 > 0, \quad d_1 > 0, \quad d_2 > 0, \quad (p_1 + q_1)d_1 - p_2^2d_2 > 0,\]
where \( d_1 = p_3d_2 - (p_1 + q_1) \), \( d_2 = p_0 + q_0 \).

The Routh–Hurwitz criterion states that all the roots of Eq. (3.10) have negative real parts, and it can be concluded that the positive equilibrium \( E^* \) (\( S^*, \ I^*, \ D^*, \ Q^* \)) is locally asymptotically stable in the absence of time delay.

Substituting \( \lambda = i\omega (\omega > 0) \) into Eq. (3.9) and separating the real and imaginary parts, the following two equations can be derived:
\[
\begin{align*}
\omega^4 - p_2\omega^2 + p_0 + q_1\omega\cos(\omega\tau) + q_0\omega\sin(\omega\tau) & = 0 \\
-p_3\omega^3 + p_1\omega + q_1\omega\cos(\omega\tau) - q_0\omega\sin(\omega\tau) & = 0.
\end{align*}
\] (3.11) (3.12)

From (3.11) and (3.12), the following equation can be obtained:
\[
q_1^2\omega^2 + q_0^2 = (\omega^4 - p_2\omega^2 + p_0)^2 + (p_3\omega^3 - p_1\omega)^2;
\]
i.e.,
\[
\omega^8 + D_3\omega^6 + D_2\omega^4 + D_1\omega^2 + D_0 = 0,
\] (3.13)
where
\[
D_3 = p_5^2 - 2p_2, \quad D_2 = p_2^2 + 2p_0 - 2p_1p_3, \\
D_1 = p_1^2 - q_1^2 - 2p_2p_0, \quad D_0 = p_0^2 - q_0^2.
\]

Let \( z = \omega^2 \). Then Eq. (3.13) can be written as
\[
h(z) = z^4 + D_2z^3 + D_2z^2 + D_1z + D_0.
\] (3.14)

Li et al. [11] obtained the following results on the distribution of roots of Eq. (3.14). Denote
\[
m = \frac{1}{2}D_2 - \frac{3}{16}D_3, \quad n = \frac{1}{32}D_2^3 - \frac{1}{8}D_1D_2 + D_1, \\
\Delta = \left( \frac{n}{2} \right)^2 + \left( \frac{m}{3} \right)^3, \quad \sigma = -1 + i\sqrt{3}, \\
y_1 = \frac{1}{\sqrt{2}}\left( -\frac{n}{2} + \sqrt{\Delta} + \sqrt{-\frac{n}{2} - \sqrt{\Delta}} \right), \\
y_2 = \frac{1}{\sqrt{2}}\left( -\frac{n}{2} + \sqrt{\Delta\sigma} + \sqrt{-\frac{n}{2} - \sqrt{\Delta\sigma^2}} \right), \\
y_3 = \frac{1}{\sqrt{2}}\left( -\frac{n}{2} + \sqrt{\Delta\sigma^2} + \sqrt{-\frac{n}{2} - \sqrt{\Delta\sigma}} \right), \\
z_i = y_i - \frac{3D_3}{4}, \quad i = 1, 2, 3.
\]

**Lemma 3.1.** For the polynomial equation (3.14):

(i) If \( D_0 = 0 \), then Eq. (3.14) has at least one positive root.
(ii) If \( D_0 \geq 0 \) and \( \Delta \geq 0 \), then Eq. (3.14) has positive roots if and only if \( z_1 > 0 \) and \( h(z_1) < 0 \).
(iii) If \( D_0 \geq 0 \) and \( \Delta < 0 \), then Eq. (3.14) has positive roots if and only if there exists at least one \( z^* \in (z_1, z_2, z_3) \), such \( z^* > 0 \) and \( h(z^*) \leq 0 \).

**Lemma 3.2.** Suppose that \( \text{(H}_2\text{)} \quad p_3 > 0, \quad d_1 > 0, \quad d_2 > 0, \quad (p_1 + q_1)d_1 - p_2^2d_2 > 0 \) is satisfied.

(i) If one of the following holds: (a) \( D_0 < 0 \); (b) \( D_0 \geq 0, \Delta \geq 0, z_1 > 0 \) and \( h(z_1) < 0 \); (c) \( D_0 \geq 0, \Delta < 0 \), and there exists at least a \( z^* \in (z_1, z_2, z_3) \) such that \( z^* > 0 \) and \( h(z^*) \leq 0 \), then all roots of Eq. (3.9) have negative real parts when \( \tau \in [0, \tau_0] \); here, \( \tau_0 \) is a certain positive constant.

(ii) If the conditions (a)–(c) of (i) are not satisfied, then all roots of Eq. (3.9) have negative real parts for all \( \tau \geq 0 \).
**Proof.** When \( \tau = 0 \), Eq. (3.9) becomes

\[
\lambda^4 + p_3 \lambda^3 + p_2 \lambda^2 + (p_1 + q_1) \lambda + (p_0 + q_0) = 0.
\]

By the Routh–Hurwitz criterion, all roots of Eq. (3.10) have negative real parts if and only if \( p_3 > 0, d_1 > 0, d_2 > 0, (p_1 + q_1)d_1 - p_2^2 d_2 > 0 \).

From **Lemma 3.1**, it can be known that, if (a)–(c) are not satisfied, then Eq. (3.9) has no roots with zero real part for all \( \tau \geq 0 \); if one of (a)–(c) holds, when \( \tau \neq \tau(0) \), \( k = 1, 2, 3, 4, j \geq 1 \), Eq. (3.9) has no roots with zero real part and \( \tau_0 \) is the minimum value of \( \tau \), so Eq. (3.9) has purely imaginary roots. According to [13], one obtains the conclusion of the lemma.

Let \( \lambda(\tau) = \nu(\tau) + i\omega(\tau) \) be the root of Eq. (3.9), \( \nu(\tau_0) = 0, \omega(\tau_0) = \omega_0 \).

From **Lemmas 3.1 and 3.2**, the following are obtained.

When conditions (a)–(c) of **Lemma 3.2(i)** are not satisfied, \( h(z) \) always has no positive root. Therefore, under these conditions, Eq. (3.9) has no purely imaginary roots for any \( \tau > 0 \), and this also implies that the positive equilibrium \( E^+(S^*, I^*, D^*, Q^*, V^*) \) of system (3.3) is absolutely stable. Thus one can easily obtain the following theorem on the stability of positive equilibrium \( E^+(S^*, I^*, D^*, Q^*, V^*) \) of system (3.3).

**Theorem 3.2.** Assume that \((H_2)\) holds. (a) \( D_0 \geq 0, \Delta \geq 0, z_1 < 0 \) or \( h(z_1) > 0 \); (b) \( D_0 \geq 0, \Delta < 0 \), and there is no \( z^* \in (z_1, z_2, z_3) \) such that \( z^* < 0 \) and \( h(z^*) \leq 0 \). Then the positive equilibrium \( E^+(S^*, I^*, D^*, Q^*, V^*) \) of system (3.3) is absolutely stable, namely, \( E^+(S^*, I^*, D^*, Q^*, V^*) \) is asymptotically stable for any time delay \( \tau \geq 0 \).

In what follows, it is assumed that the coefficients in \( h(z) \) satisfy the following condition.

\[ (H_3) \quad (a) \ D_0 \geq 0, \Delta \geq 0, z_1 < 0 \text{ or } h(z_1) > 0; \quad (b) \ D_0 \geq 0, \Delta < 0, \text{ and there is no } z^* \in (z_1, z_2, z_3) \text{ such that } z^* < 0 \text{ and } h(z^*) \leq 0. \]

Then, according to a lemma in [14], it is known that Eq. (3.14) has at least a positive root \( \omega_0 \), which means that the characteristic equation (3.9) has a pair of purely imaginary roots \( \pm i\omega_0 \).

Since Eq. (3.9) has a pair of purely imaginary roots \( \pm i\omega_0 \), the corresponding \( \tau_0 > 0 \) are given by (3.11) and (3.12).

\[
\tau_k = \frac{1}{\omega_0} \arccos \left[ \frac{q_0(2p_2^2 - \omega_0^2 - p_0) + q_1(\omega_0(p_3\omega_0^3 - p_1\omega_0))}{q_1^2 \omega_0^2 + q_0^2} \right] + \frac{2k\pi}{\omega_0} \quad (k = 0, 1, 2, \ldots).
\]

Let \( \lambda(\tau) = \nu(\tau) + i\omega(\tau) \) be the roots of Eq. (3.9) such that when \( \tau = \tau_k, \nu(\tau_k) = 0 \) and \( \omega(\tau_k) = \omega_0 \) are true.

**Lemma 3.3.** Suppose that \( h'(z_0) \neq 0 \). If \( \tau = \tau_0 \), then \( \pm i\omega_0 \) is a pair of simple purely imaginary roots of Eq. (3.9). Moreover, if the conditions of **Lemma 3.2(i)** are satisfied, then \( \frac{\text{d} \text{Re}\lambda(\tau)}{\text{d}\tau} > 0 \).

It can be claimed that

\[
\text{sgn} \left[ \frac{\text{d} \text{Re}\lambda}{\text{d}\tau} \right]_{\tau = \tau_k} = \text{sgn}[h'(\omega_0^2)].
\]

This will signify that there exists at least one eigenvalue with positive real part for \( \tau > \tau_k \). Differentiating Eq. (3.9) with respect to \( \tau \), one obtains

\[
\left( \frac{d\lambda}{d\tau} \right)^{-1} = \frac{(4\lambda^3 + 3p_3\lambda^2 + 2p_2\lambda + p_1)q_1e^{-\lambda t} - (q_1\lambda + q_0)\tau e^{-\lambda t}}{(q_1\lambda + q_0)\lambda e^{-\lambda t}} = \frac{(4\lambda^3 + 3p_3\lambda^2 + 2p_2\lambda + p_1)e^{\lambda t}}{(q_1\lambda + q_0)\lambda} + \frac{q_1}{(q_1\lambda + q_0)\lambda} - \frac{\tau}{\lambda}.
\]

Therefore

\[
\text{sgn} \left[ \frac{\text{d} \text{Re}\lambda}{\text{d}\tau} \right]_{\tau = \tau_k} = \text{sgn} \left[ \text{Re} \left( \frac{d\lambda}{d\tau} \right)^{-1} \right] \bigg|_{\lambda = i\omega_0} = \text{sgn} \left[ \text{Re} \left( \frac{(4\lambda^3 + 3p_3\lambda^2 + 2p_2\lambda + p_1)e^{\lambda t}}{(q_1\lambda + q_0)\lambda} + \frac{q_1}{(q_1\lambda + q_0)\lambda} - \frac{\tau}{\lambda} \right) \right] \bigg|_{\lambda = i\omega_0}
\]

\[
= \text{sgn} \text{Re} \left\{ \left( -4\omega_0^2 i - 3p_3\omega_0^2 + 2p_2\omega_0 + p_1 \right) \frac{\cos(\omega_0 \tau_k) + i \sin(\omega_0 \tau_k)}{q_1 \omega_0 + q_0} + \frac{q_1}{q_1 \omega_0 + q_0} \right\}
\]

\[
= \frac{1}{\tau_k} \left\{ 2\omega_0 \left[ (2p_2\omega_0 - 4\omega_0^3) \sin(\omega_0 \tau_k) + (3p_2\omega_0^2 - p_1) \cos(\omega_0 \tau_k) \right] q_1 \omega_0^2 + \left[ (2p_2\omega_0 - 4\omega_0^3) \cos(\omega_0 \tau_k) - (3p_2\omega_0^2 - p_1) \sin(\omega_0 \tau_k) \right] q_0 \omega_0 - q_1^2 \omega_0^2 \right\}.
\]
Suppose that conditions (3.3) give the four curves of infected hosts in four coordinates in 30 min. In the initial stage of worm propagation, the timedelay has a little effect on the number of infected hosts with four delays, \( \tau \), and the number of infected hosts is influenced by every different value of the delay. Long numerical simulation in experiments, which can be mainly divided into two parts. We could compare the behavior of worm propagation in anomaly detection with time delay by worms which have different scan rates. In order to verify that bifurcation exists in the worm propagation model with time delay, we carried out several experiments, which can be mainly divided into two parts.

4. Numerical simulations

In order to simulate the real behavior of the spread of a worm, the parameters in the experiments are practical values for when worms break out in real life. The Slammer worm and the Code Red worm were selected for experiments, so that we could compare the behavior of worm propagation in anomaly detection with time delay by worms which have different scan rates. In order to verify that bifurcation exists in the worm propagation model with time delay, we carried out several experiments, which can be mainly divided into two parts.

4.1. Numerical simulation of the Slammer worm

The vulnerable population \( N \) is set to 750,000. According the real conditions of the Slammer worm, the worm's average scan rate is \( \eta = 4000 \) per second. The worm's infection rate can then be calculated as \( \alpha = \eta N/2^{32} = 0.0698 \) [7]. This means that an average 0.0698 hosts of all hosts can be scanned by one host. The infection ratio is \( \beta = \eta/2^{32} = 0.00000093 \). The immunization rate of infected hosts is set to \( \gamma = 0.01 \), and the immunization rate of quarantined hosts \( \delta = 0.005 \). The reassembly rate of immunization hosts is \( \mu = 0.08 \). At the beginning, there are 10 infected hosts, while others are susceptible.

In the anomaly detection system, the rate at which infected hosts are detected and quarantined is \( \alpha_2 = 0.2 \) per second. This means that an infected host can be detected and quarantined in about 5 s. The rate at which susceptible hosts are detected and quarantined is \( \alpha_1 = 0.00002315 \) per second, i.e., about two false alarms are generated by the anomaly detection system per day.

When \( \tau = 5 < \tau_0 \), Fig. 4 shows the changes of the numbers of five kinds of host in 5 min. By Theorem 3.3, the equilibrium \( E^*(S^*, I^*, D^*, Q^*, V^*) \) is asymptotically stable when \( \tau \in (0, \tau_0) \); this property is illustrated by the numerical simulation in Fig. 4. Every kind of host will be stable. When \( \tau \) passes through the critical value \( \tau_0 \), the worm \( E^* \) will lose its stability and bifurcate, where there is a family of periodic solution bifurcates from equilibrium \( E^* \), which is depicted by the one hour long numerical simulation in Fig. 5.

With other parameters remaining the same, the delay \( \tau \) is set to a different value each time in order to state that the number of infected hosts is influenced by every different value of the delay. Fig. 6 gives four curves of the number of infected hosts with four delays, \( \tau = 5, \tau = 15, \tau = 45, \) and \( \tau = 60 \), respectively, in 10 min. Fig. 7 gives the four curves of infected hosts in four coordinates in 30 min. In the initial stage of worm propagation, the time delay has a little effect on the number of detected and quarantined infected hosts in different delays and in different coordinates.
infected hosts; the initial stages of the four curves are overlapping. With an increase of the time $T$, the time delay affects the number of infected hosts. With the increase of the time delay $\tau$, it will exceed the value of $\tau_0$ at which the worm propagation will become unstable, which verifies our theory.

Fig. 8(a) shows a projection of the phase portrait of system (3.3) as $\tau = 40 < \tau_0$ in ($S, I, V$)-space. Fig. 8(b) shows a projection of the phase portrait of system (3.3) as $\tau = 60 > \tau_0$ in ($S, I, V$)-space. Fig. 9(a) and (b) show the phase portraits of susceptible hosts $S(t)$ and infected hosts $I(t)$ as $\tau = 40$ and $\tau = 60$, respectively.

4.2. The numerical simulation of the Code Red worm

According to the real conditions of the Code Red worm, it is assumed that the vulnerable population is $N = 1,000,000$ and the worm’s average scan rate is $\eta = 358$ per minute. The rest of parameters are the same as for the Slammer worm. The results are shown in Figs. 10–14.

The propagation trend of the Code Red worm is similar to that of the Slammer worm. In Figs. 10 and 11, when the time delay $\tau$ is less than $\tau_0$, the system is stable, and the Code Red worm can be easily restrained; when the time delay $\tau$ is equal to or greater than $\tau_0$, a Hopf bifurcation appears, which means that the system loses its stability and worm propagation is out of control. Fig. 12 gives the changing trend of the number of infected hosts with the increase of $\tau$, which illustrates
Fig. 7. The number of infected hosts $I(t)$ when $\tau$ is changed in four figures.

Fig. 8. The projection of the phase portrait of system (3.3) in $(S, I, V)$-space.

Fig. 9. The phase portrait of susceptible hosts $S(t)$ and infected hosts $I(t)$.

Fig. 10. Comparison of the five kinds of host if $\tau < \tau_0$.  

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Fig. 11. Comparison of the three kinds of host if $\tau > \tau_0$.

Fig. 12. The number of infected hosts $I(t)$ when $\tau$ is changed.

Fig. 13. The projection of the phase portrait of system (3.3) in $(S, I, V)$-space.

Fig. 14. The phase portrait of susceptible hosts $S(t)$ and infected hosts $I(t)$.

the changing rule from stability to instability of the worm propagation system. Figs. 13 and 14 denote the projection of the phase portraits of system (3.3) in $(S, I, V)$-space and the $(S, I)$-plane.

5. Conclusion

A quarantine strategy fulfills its function in worm containment and is widely used. The implementation of quarantine depends on the IDS. For an intrusion detection system which requires a low false positive rate, worm detection takes some
time, called the time delay, which is influenced by the size of the time window. In order to accord with actual facts, a worm propagation model with time delay under quarantine defense is constructed. Moreover, time delay may lead to Hopf bifurcation and make the worm propagation system unstable. Thus, the analysis of Hopf bifurcation can help ensure that the worm propagation system is stable and can help in the elimination of Internet worms.

In this paper, on this basis of the SEM and the SIVS model, a quarantine strategy is introduced to construct an SIDQV model in which time delay is considered. Next, the stability of the positive equilibrium and the local Hopf bifurcation under this model are analyzed. By theoretical analysis, the following conclusions can be derived.

- The critical time delay $\tau_0$ where the Hopf bifurcation appears is obtained. $\tau_0 = \frac{1}{\omega_0} \arccos \left[ \frac{q_0(p_2\omega_0^2 - \omega_0^4 - p_0) + q_1p_0(p_2\omega_0^2 - p_1\omega_0)}{q_1^2\omega_0^6 + q_0^2} \right]$.
- When the time delay $\tau < \tau_0$, the worm propagation system is stable. In such conditions, the characteristics of worm propagation can be easily predicted, and Internet worms can get eliminated.
- When the time delay $\tau \geq \tau_0$, Hopf bifurcation occurs, which implies that the worm propagation system is unstable and out of control.

In practice, the stability of the worm propagation system must be guaranteed. In order to predict and even eliminate worm propagation, the time delay $\tau$ should remain less than $\tau_0$ by decreasing the window size of the IDS.

Various factors in the network can affect worm propagation. Among them, time delay is important, but has been studied less. The paper thus concentrates on analyzing it. Of course, other network impacts of worms cannot be neglected, and they will be a major emphasis of our future research.

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